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CONTENTS

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AN ELEMENTARY PROOF OF JACKSON'S THEOREM ON MEAN-APPROXIMATION

E. W. CHENEY, University of Texas, Austin and University of California, Los Angeles

In 1921 Dunham Jackson proved that every continuous function on the interval $[a, b]$ possesses a *unique* best approximation-in-the-mean by polynomials of degree $\leqq n$ [**1**; proof reproduced in **8**, p. 76]. In other words, corresponding to each $f \in C[a, b]$ and to each natural number n there is a polynomial p of degree $\leqq n$ such that for all other polynomials q of degree $\leqq n$,

$$\int |f - p| < \int |f - q|.$$

The existence part of this theorem is of relatively little interest because F. Riesz has proved that in *any* normed linear space a finite dimensional subspace must contain a point closest to each outside point [**2**; see also **8**, p. 10]. *The unicity part of Jackson's theorem is, however, of considerable interest.* In the first place, it seems to require a tailor-made argument, since the only known *general* theorem on unicity hypothesizes strict convexity—a property *not* possessed by the norm $\int|f|$. (See [**8**, p. 11].) In the second place, the extant proofs [**1**, **3**, **4**, **5**, **6**] of Jackson's theorem entail some measure theory (albeit of an elementary nature) or some functional analysis, whereas the *statement* of the theorem involves nothing but continuous functions and Riemann integrals. We now give this theorem an elementary proof, which is completely eclectic, borrowing its ideas from the proofs of Walsh and Motzkin, Pták, and Kreĭn.

The theorem of Jackson remains true if the linear space of polynomials having degree $\leqq n$ is replaced by *any* finite dimensional Haar subspace. An n-dimensional subspace P in $C[a, b]$ is called a *Haar* subspace if 0 is the only element of P which has n (or more) roots in $[a, b]$. Kreĭn was the first to extend Jackson's theorem in this way [**6**].

In what follows, we designate by *sgn f* the function whose values are

$$(\operatorname{sgn} f)(x) = \begin{cases} +1 & f(x) > 0 \\ -1 & f(x) < 0 \\ 0 & f(x) = 0. \end{cases}$$

If f is continuous and possesses at most a finite number of roots, then sgn f is Riemann integrable.

LEMMA. *Let f and h be elements of $C[a, b]$. Assume that f has finitely many roots and that $\int h$ sgn $f \neq 0$. Then for some real λ, $\int|f-\lambda h| < \int|f|$.*

Proof. Let $x_1, \cdots, x_k$ be all the roots of f lying in the open interval (a, b). For small positive ϵ, the set

$$A = [a+\epsilon, x_1-\epsilon] \cup [x_1+\epsilon, x_2-\epsilon] \cup \cdots \cup [x_k+\epsilon, b-\epsilon]$$

consists of $k+1$ nondegenerate closed intervals. Let B denote the complement of A in $[a, b]$. For definiteness, and without loss of generality, we may assume that $\int h \operatorname{sgn} f > 0$. Select ϵ small enough to ensure the inequality

$$\int_A h \operatorname{sgn} f > \int_B |h|.$$

This is possible because the intervals comprising B have a total length $(2k+1)\epsilon$, and the function $|h|$ is bounded. Since A is closed and contains no roots of f, the number $m = \min\{|f(x)| : x \in A\}$ is positive. If λ is chosen such that $0 < \lambda \max|h(x)| < m$, then for points in A, $|\lambda h(x)| < m \leqq |f(x)|$, and consequently (on A) $\operatorname{sgn} f = \operatorname{sgn}(f - \lambda h)$. Thus we have

$$\begin{aligned}
\int |f - \lambda h| &= \int_B |f - \lambda h| + \int_A |f - \lambda h| \\
&= \int_B |f - \lambda h| + \int_A (f - \lambda h) \operatorname{sgn} f \\
&= \int_B |f - \lambda h| + \int_A |f| - \lambda \int_A h \operatorname{sgn} f \\
&= \int_B |f - \lambda h| - \int_B |f| + \int |f| - \lambda \int_A h \operatorname{sgn} f \\
&\leqq \lambda \int_B |h| - \lambda \int_A h \operatorname{sgn} f + \int |f| \\
&< \int |f|.
\end{aligned}$$

JACKSON'S THEOREM. *Let P be a Haar subspace of $C[a, b]$. Then each f in $C[a, b]$ possesses at most one best approximation-in-the-mean from P.*

Proof. Suppose that f has two best approximations, p_1 and p_2, in P. Because of the inequality

$$\int |f - \tfrac{1}{2}(p_1 + p_2)| \leqq \tfrac{1}{2} \int |f - p_1| + \tfrac{1}{2} \int |f - p_2|$$

the function $\frac{1}{2}(p_1 + p_2)$ is also a best approximation of f. Hence

$$\int (|f - p| - \tfrac{1}{2}|f - p_1| - \tfrac{1}{2}|f - p_2|) = 0.$$

Since the integrand is continuous and $\leqq 0$, it must vanish identically. Now let n denote the dimension of P. If $f - p$ has n (or more) roots, then because of the equality $|f - p| = \frac{1}{2}|f - p_1| + \frac{1}{2}|f - p_2|$, we may conclude that $f - p_1$, $f - p_2$, and $p_1 - p_2$ must have the same n roots. Thus, by the Haar property, $p_1 = p_2$.

Let us assume therefore that the function $f_0 \equiv f - p$ has *at most* $n - 1$ roots. Then there exist points $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ containing among them all roots of f_0. By the lemma, the expression

$$\int h \operatorname{sgn} f_0 = \sum_{i=1}^{n} \sigma_i \int_{x_{i-1}}^{x_i} h = \sum_{i=1}^{n} \sigma_i \phi_i(h) \qquad (\sigma_i = \pm 1)$$

vanishes for each $h \in P$, because otherwise one could secure the inequality $\int |f_0 - \lambda h| < \int |f_0|$ with an appropriate choice of $h \in P$ and λ. If $\{g_1, \cdots, g_n\}$ is a basis for P then the matrix $[\phi_i(g_j)]$ is singular, because $\sum_i \sigma_i \phi_i(g_j) = 0$. Thus we can find a nonzero n-tuple $(c_1, \cdots, c_n)$ such that $\sum_j c_j \phi_i(g_j) = 0$. This equation implies that the nonzero function $h = \sum_j c_j g_j$ has the property

$$\int_{x_{i-1}}^{x_i} h = \phi_i(h) = \phi_i\left(\sum_j c_j g_j\right) = \sum_j c_j \phi_i(g_j) = 0.$$

But then h must possess at least one root in each interval (x_{i-1}, x_i), in contradiction with the Haar property.

The preparation of this paper was supported by the Air Force Office of Scientific Research.

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1. D. Jackson, Note on a class of polynomials of approximation, Trans. Amer. Math. Soc., 22 (1921) 320–326.
2. F. Riesz, Über lineare Funktionalgleichungen, Acta Math., 41 (1918) 71–98.
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LIMIT OF A FUNCTION AND A CARD TRICK

ALI R. AMIR-MOÉZ, Texas Technological College

This article intends to explain some mathematical ideas which cause a card trick to work. We believe there is some pedagogical value in this note in regard to the concept of limit.

Choose 21 cards and put them down, face up, one at a time, in three sets. We shall explain the idea more thoroughly. Put down three cards first and add one card to each. Continue in this manner until three sets of equal number of cards are obtained. While doing this we ask a friend to choose a card and to tell us only in which set the chosen card is. Then we put the set, in which the chosen card is, between the other two sets. We repeat the same thing and ask all roots of f_0. By the lemma, the expression

$$\int h \operatorname{sgn} f_0 \equiv \sum_{i=1}^{n} \sigma_i \int_{x_{i-1}}^{x_i} h \equiv \sum_{i=1}^{n} \sigma_i \phi_i(h) \qquad (\sigma_i = \pm 1)$$

vanishes for each $h \in P$, because otherwise one could secure the inequality $\int |f_0 - \lambda h| < \int |f_0|$ with an appropriate choice of $h \in P$ and λ. If $\{g_1, \cdots, g_n\}$ is a basis for P then the matrix $[\phi_i(g_j)]$ is singular, because $\sum_i \sigma_i \phi_i(g_j) = 0$. Thus we can find a nonzero n-tuple $(c_1, \cdots, c_n)$ such that $\sum_j c_j \phi_i(g_j) = 0$. This equation implies that the nonzero function $h = \sum_j c_j g_j$ has the property

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But then h must possess at least one root in each interval (x_{i-1}, x_i), in contradiction with the Haar property.

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Choose 21 cards and put them down, face up, one at a time, in three sets. We shall explain the idea more thoroughly. Put down three cards first and add one card to each. Continue in this manner until three sets of equal number of cards are obtained. While doing this we ask a friend to choose a card and to tell us only in which set the chosen card is. Then we put the set, in which the chosen card is, between the other two sets. We repeat the same thing and ask our friend again to tell us in which set the chosen card is. Again we put this set between the other two sets. The third time we sort the cards into three sets as before and we observe that the chosen card is exactly at the middle of the set in which the chosen card has gone.

The reader may have already seen this trick or knows it. We are not trying to teach a card trick. We would like to study the mathematical explanation of the trick and look into some generalizations of it.

Let us say that the position of the chosen card is a function of the number of times we have gone through the process of putting cards into three sets. In this particular case we observe that $8 \leqq f(1) \leqq 14$. This is quite obvious since the chosen card is in the set of seven cards which is in between the other two sets of seven cards.

Now by letting $f(1)=14$ or $f(1)=8$ we shall find that $10 \leqq f(2) \leqq 12$. Indeed this is easily done for a set of 21 cards. We shall obtain a technique of finding the bounds of $f(k)$ for the general case.

Next we see that $11 \leqq f(3) \leqq 11$. Thus $f(3)=11$. We may say

$$\lim_{x \to 3} f(x) = 11.$$

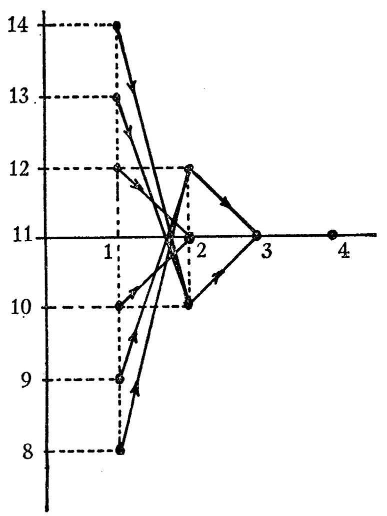


FIG. 1.

We shall draw a diagram giving the bounds of $f(x)$ for each x (Fig. 1). We should really be less sloppy with notations. For each x the set $\{f(x)\}$ is a bounded set, both above and below. Thus we should use the ideas of least upper bounds and greatest lower bounds which, in fact, belong to the set $\{f(x)\}$ in this problem. The diagram shows the least upper bound and greatest lower bound of $\{f(x)\}$ for $x=1$, 2, and 3. We also use an arrow in the diagram to show:

For $f(1)=14$ or $f(1)=13$ we get $f(2)=10$. But if $f(1)=12$ then $f(2)=11$.

For $f(1)=8$ or $f(1)=9$ we get $f(2)=12$. But if $f(1)=10$, then $f(2)=11$.

Thus there are several ways that $f(x)$ approaches its limit.

Now let us look into an example where the limit does not exist. Chose 24 cards and do the same thing with them, i.e., set them into three sets and choose a card. Then we put the set in which the chosen card appears, between the other two. We repeat as many times as we would like. We observe that

$$9 \leqq f(1) \leqq 16,$$
$$11 \leqq f(2) \leqq 14,$$
$$12 \leqq f(3) \leqq 13,$$
$$12 \leqq f(4) \leqq 13,$$
$$\cdots\cdots\cdot.$$

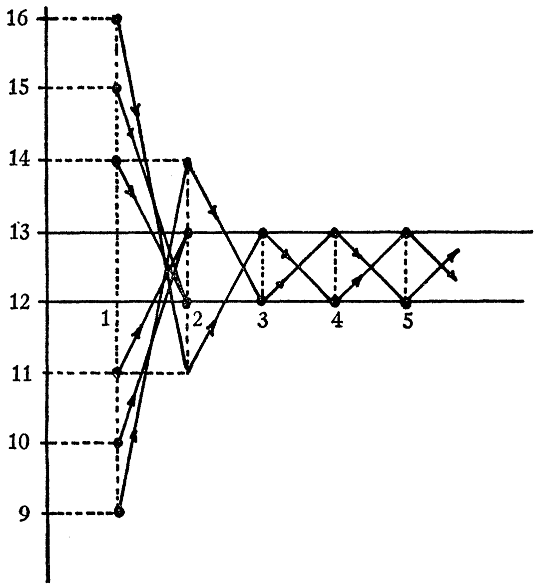


FIG. 2.

Thus the least upper bound and greatest lower bound of any set $\{f(x)\}$ never become the same. Here again we supply a diagram (Fig. 2). The diagram illustrates the following observation:

For $f(1)=16$ we have $f(2)=11$. But for $f(1)=15$ or $f(1)=14$ we have $f(2)=12$.

For $f(1)=9$ we have $f(2)=14$. But for $f(1)=10$ or $f(1)=11$ we have $f(2)=13$. Indeed we can also do a card trick in this case. After setting the cards down three times, we know that the chosen card will be in the position 12 or 13. In this way we keep the two possible cards in mind. In the next sorting of the cards we can choose the correct one.

Now let us look into a more general case. In what follows every small letter denotes a positive integer. Let the number of cards be $3q$. Then we look for a number k such that $\lim_{x\to k} f(x)$ exists, where x is defined as before. Here k indicates the number of times which is necessary and perhaps sufficient to put the cards into three sets in order to get the chosen card in the middle. It is clear that $q+1 \leqq f(1) \leqq 2q$.

Consider the case $f(1)=2q$. We have to divide the cards into three sets as was described. We know that $2q=3c+r$, where $0\leqq r<3$. But we shall write instead

(1) $$2q = 3c_2 - r_2, \qquad 0 \leqq r_2 < 3.$$

The reader will discover the advantage of this choice as we proceed. We observe that (1) implies that

(2) $$q - r_2 = 3m_2.$$

The equalities (1) and (2) show that first we put down three sets of c_2 cards and three sets of m_2 cards on the top of them (Fig. 3). Each set is indicated by a rectangle and the number of cards are written inside the rectangle. The equality (1) shows that the chosen card is on the top of one of the sets of c_2 cards. Thus in this case

(3) $$f(2) = q + 1 + m_2.$$

Now let $f(1)=q+1$. Then by (2) we have

$$q + 1 = 3m_2 + r_2 + 1, \qquad 0 \leqq r_2 < 3.$$

In this case we first put down three sets of m_2 cards each (Fig. 4). Then we observe that the rest of the cards, i.e.,

$$3q - 3m_2 = 3q - (q - r_2) = 2q + r_2$$

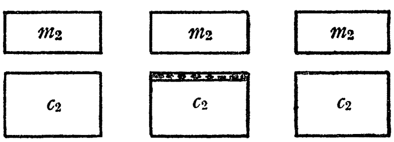


FIG. 3.

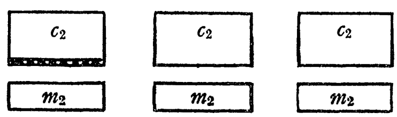


FIG. 4.

is divisible by 3; in fact $2q+r_2=3c_2$.

Thus a set of c_2 cards is put over each set of m_2 cards. The chosen card is at the bottom of one of these sets. Thus $f(2)=q+c_2$. But (1) and (2) imply that $3q=3c_2+3m_2$ or $q=c_2+m_2$. Therefore

(4) $$f(2) = 2q - m_2.$$

The values of $f(2)$ in (3) and (4) are, respectively, the greatest lower bound and the least upper bound of the set $\{f(2)\}$. Thus $q+1+m_2\leqq f(2)\leqq 2q-m_2$, where

(5) $$q = 3m_2 + r_2, \qquad 0 \leqq r_2 \leqq 3.$$

The reader may easily show that $q+1+m_2 \leqq 2q-m_2$. The equality holds if and only if $q=1$.

Now suppose that in a similar way as to what was described so far we obtain

$$q + 1 + m_{k-1} \leqq f(k-1) \leqq 2q - m_{k-1},$$

where $q+m_{k-2}=3m_{k-1}+r_{k-1}$, $0 \leqq r_{k-1} < 3$ and

$$q + 1 + m_{k-1} < 2q - m_{k-1}.$$

Let us for simplicity write $a \leqq f(k-1) \leqq b$. As before let $f(k-1)=b$. Then $b=3c_k-r_k$ where $0 \leqq r_k < 3$. This implies that

$$3q - b - r_k = 3m_k. \tag{6}$$

Again we observe that putting the cards into three sets has two stages. First three sets of c_k cards are put down (Fig. 5). Then three sets of m_k cards are put over them. As before we observe that the chosen card is at the top of one of the sets of c_k cards. This implies that the greatest lower bound of $\{f(k)\}$ is $q+1+m_k$.

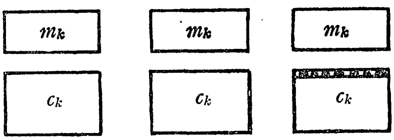


FIG. 5.

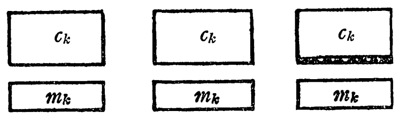


FIG. 6.

Now let $f(k-1)=a$. Then we observe that $a+b=3q+1$ or $a=3q-b+1$. This and (6) imply that $3q-b=3m_k+r_k$. Therefore $a-1=3m_k+r_k$ where $0 \leqq r_k < 3$. Here again we put down three sets of m_k cards and a set of c_k cards on top of each of them (Fig. 6). The chosen card is at the bottom of one of the sets which has c_k cards. Thus $f(k)=q+c_k$.

Since $q=c_k+m_k$, it follows that $f(k)=2q-m_k$ and this is the least upper bound of $\{f(k)\}$. Thus we get $q+1+m_k \leqq f(k) \leqq 2q-m_k$ where

$$q + m_{k-1} = 3m_k + r_k, \qquad 0 \leqq r_k < 3.$$

This gives a technique of getting $f(k)$ from $f(k-1)$. The reader may show that $m_k \geqq m_{k-1}$ and $2q-m_k \geqq q+1+m_k$. Let us study the case that $q=2h+1$. Here the inequality $q+1+m_k \leqq 2q-m_k$ implies that $m_k \leqq h$. The equality holds if and only if $m_k=h$. But, in general, we have $m_{k+1} \geqq m_k$. Thus $q+1+m_k=2q-m_k$ if and only if $m_{k+1}=m_k=h$. Then this k is the number of times necessary and sufficient to put the cards into three sets in order to have $f(k)=q+h+1$.

In the case where $q=2h$ a necessary and sufficient condition for $q+2+m_k = 2q-m_k$ is that $m_k=m_{k-1}=h-1$. We omit the proof since it is very similar to the previous case.

It would be interesting if an explicit formula for k in terms of h could be obtained. We leave it as a problem.

The reader might consider the generalization to the cases of putting a set of cards into 5, 7, $\cdots$, $2h+1$ sets. Are there other generalizations?

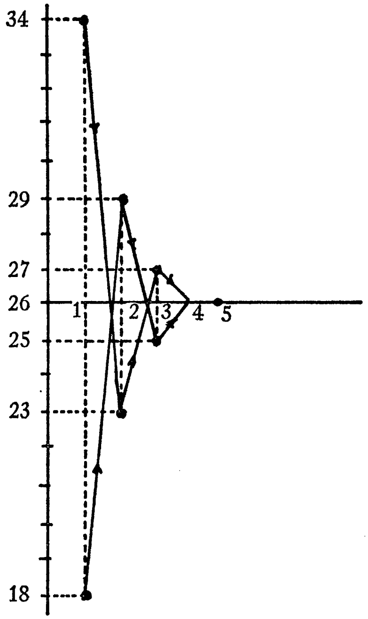


FIG. 7.

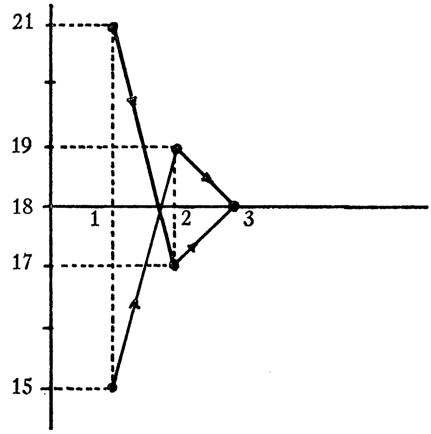


FIG. 8.

We conclude this article by giving diagrams for a set of 51 cards (Fig. 7) and putting them into 3 sets, and 35 cards and putting them into 5 sets (Fig. 8). We observe that k for 51 cards and 3 sets is 4. We also see that k for 35 cards and 5 sets is 2.

A FUNCTION WHOSE VALUES ARE INTEGERS

JOSEPH ARKIN, Suffern, New York

Introduction. In this paper, using the determinants of certain triangular matrices, we prove the following

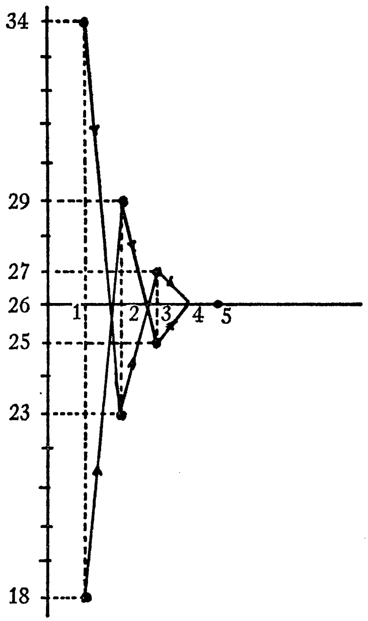
THEOREM. *For any positive integers k and n,*

$$(1)\qquad ((2k+1)!/3\cdot 2^k k!)\left(\sum_{r=1}^{n} r^{2k} \Big/ \sum_{r=1}^{n} r^2\right)$$

is an integer.

Proof. For any integers k and w, $k>0$, notice that

$$(2)\qquad \text{(a)}\quad (w+1)^{2k+1} - w^{2k+1} = 1 + \sum_{r=1}^{2k} \binom{2k+1}{r} w^{2k+1-r},$$



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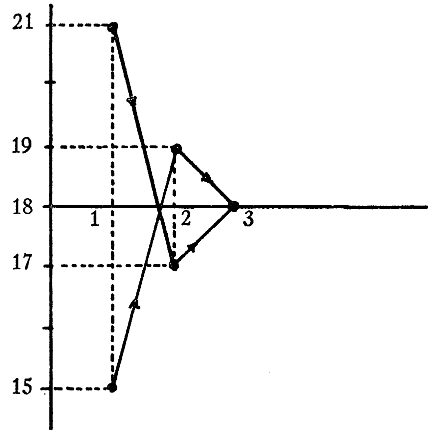

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$$(2) \qquad \text{(b)} \quad (w-1)^{2k+1} - w^{2k+1} = -1 + \sum_{r=1}^{2k} (-1)^r \binom{2k+1}{r} w^{2k+1-r},$$

where

$$\binom{p}{q} = \frac{p!}{q!(p-q)!} \qquad (q = 0, 1, \cdots, p).$$

For any positive integers m and n, let $S(m, n) = 1^m + 2^m + \cdots + n^m$. If in (2a) we let $w = 0, 1, 2, \cdots, n$, and add, we obtain

$$(3) \qquad (n+1)^{2k+1} = \left(\sum_{r=1}^{2k} \binom{2k+1}{r} S(2k+1-r, n) \right) + n + 1.$$

Just as we found (3) from (2a), from (2b) we get

$$(4) \qquad -n^{2k+1} = \left(\sum_{r=1}^{2k} (-1)^r \binom{2k+1}{r} S(2k+1-r, n) \right) - n.$$

Replacing k by j, say, and subtracting (4) from (3), we obtain

$$(5) \qquad a_{2j+1} = 2 \sum_{r=1}^{j} \binom{2j+1}{2r-1} S(2j+2-2r, n),$$

where

$$(6) \qquad a_{2j+1} = (n+1)^{2j+1} + n^{2j+1} - (2n+1).$$

Next we consider the k equations obtained from (5) by successively replacing j by $1, 2, \cdots, k$. With $l = 1, 2, \cdots, k$, these k equations in the k unknowns $2S(2l, n)$ can be solved by Cramer's rule to obtain

$$(7) \qquad 2(D_1)S(2k, n) = (-1)^{k+1} D_2,$$

where D_1 and D_2 are the determinants given below:

$$D_1 = \begin{vmatrix} \binom{2k+1}{1} & \binom{2k+1}{3} & \cdots & \binom{2k+1}{2k-3} & \binom{2k+1}{2k-1} \\ 0 & \binom{2k-1}{1} & \cdots & \binom{2k-1}{2k-5} & \binom{2k-1}{2k-3} \\ 0 & 0 & \cdots & \binom{2k-3}{2k-7} & \binom{2k-3}{2k-5} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & \binom{5}{1} & \binom{5}{3} \\ 0 & 0 & \cdots & 0 & \binom{3}{1} \end{vmatrix},$$

$$D_2 = \begin{vmatrix} a_{2k+1} & \binom{2k+1}{3} & \cdots & \binom{2k+1}{2k-3} & \binom{2k+1}{2k-1} \\ a_{2k-1} & \binom{2k-1}{1} & \cdots & \binom{2k-1}{2k-5} & \binom{2k-1}{2k-3} \\ a_{2k-3} & 0 & \cdots & \binom{2k-3}{2k-7} & \binom{2k-3}{2k-5} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ a_5 & 0 & \cdots & \binom{5}{1} & \binom{5}{3} \\ a_3 & 0 & \cdots & 0 & \binom{3}{1} \end{vmatrix}$$

(*Note.* We have $(-1)^{k+1}$ in the right hand side of equation (7) because, when k is even, an odd number of row and column interchanges are required to convert the determinant to the form D_2.)

Since D_1 is the determinant of a triangular matrix, it is simply the product of the diagonal entries. Hence

$$D_1 = \prod_{r=1}^{k} (2r+1) = (2k+1)!/2^k k!.$$

Equation (7) then becomes

$$(8) \qquad 6((2k+1)!/3\cdot 2^k\cdot k!)S(2k, n) = D_2.$$

To complete the proof of the theorem it suffices, therefore, to show that D_2 is a multiple of $6S(2, n)$. This will be so if every entry in a column of D_2 is such a multiple.

To this end let

$$u = 6S(2, n) = n(n+1)(2n+1).$$

Then we show that for any positive integer j

$$(9) \qquad a_{2j+1} \equiv 0 \pmod{u},$$

which shows that u divides every entry in the first column of D_2. To reduce the problem still further, notice that n, $n+1$, and $2n+1$ are relatively prime in pairs, so that the congruence (9) is equivalent to the three congruences

$$a_{2j+1} \equiv 0 \pmod{m} \qquad (m = n, n+1, 2n+1).$$

Although our notation has hidden the fact that a_{2j+1} depends on n, equation (6) shows that these congruences become

$$(n+1)^{2k+1} \equiv 1 \pmod{n},$$

$$(n+1)^{2k+1} + n^{2k+1} \equiv 0 \pmod{2n+1},$$

$$n^{2k+1} \equiv n \pmod{n+1},$$

which are easily verified. This completes the proof of the theorem.

Although there are many special cases to this theorem, the one obtained by setting $n+1=2^q m$, may be worth mentioning.

COROLLARY. *For any positive integers q, m and k,*

$$1^{2k}+\cdots+(2^q m-1)^{2k}\equiv 0 \pmod{2^{q-1}}.$$

SOME RADICAL AXES ASSOCIATED WITH THE CIRCUMCIRCLE

D. MOODY BAILEY, Princeton, West Virginia

PART I

D and D' are two points on side BC of triangle ABC. An arbitrary circle (B, C) passes through vertices B and C of the given triangle and in similar fashion a second arbitrary circle (D, D') passes through points D and D'. Circles (B, C) and (D, D') may have two points in common, a single point in common, or may be nonintersecting circles. Let the radical axis of the two circles meet side BC at point M. This point then has equal power with respect to the two circles and this means that $MB\cdot MC=MD\cdot MD'$. We observe (Fig. 1) that this equality may be written as $MB(MB+BD+DC)=(MB+BD)(MB+BD')$. A solution of this equation shows that $MB=(BD\cdot BD')/(DC-BD')$.

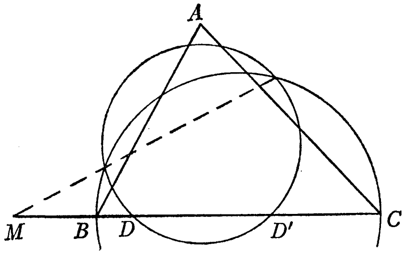

FIG. 1.

Ratio MB/MC is now formed and we write

$$MB/MC=(MB)/(MB+BD+DC).$$

A substitution of the value of MB, given at the end of the preceding paragraph, in the right member of this equation gives

$$\frac{MB}{MC}=\frac{BD\cdot BD'}{BD\cdot BD'+(BD+DC)(DC-BD')}=\frac{BD\cdot BD'}{DC(BD+DC-BD')}$$
$$=\frac{BD\cdot BD'}{DC(BC-BD')}=\frac{BD}{DC}\cdot\frac{BD'}{D'C}.$$

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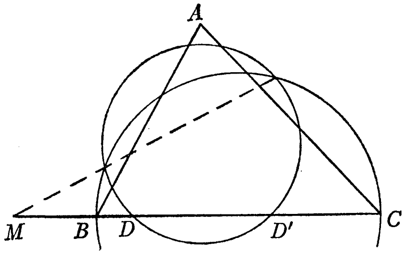

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$$\frac{MB}{MC}=\frac{BD\cdot BD'}{BD\cdot BD'+(BD+DC)(DC-BD')}=\frac{BD\cdot BD'}{DC(BD+DC-BD')}$$
$$=\frac{BD\cdot BD'}{DC(BC-BD')}=\frac{BD}{DC}\cdot\frac{BD'}{D'C}.$$

Since directed segment MB is equivalent to $-BM$, we may write $MB/MC = (BD/DC)(BD'/D'C)$ as $BM/MC = -(BD/DC)(BD'/D'C)$. The same relationship may be shown to hold when point M lies between B and C, or to the right of point C. This fundamental result then follows:

THEOREM 1. *Let D and D' be any two points on side BC of triangle ABC. An arbitrary circle passes through vertices B and C of the given triangle while a second such circle passes through points D and D'. If the radical axis of these two circles meets BC at point M, then $BM/MC = -(BD/DC)(BD'/D'C)$.*

Similar constructions may evidently be made with respect to sides CA and AB when pairs of points (E, E') and (F, F') are given on these respective sides. The radical axis of circles (C, A) and (E, E') meets CA at point N so that $CN/NA = -(CE/EA)(CE'/E'A)$. Similarly the radical axis of circles (A, B) and (F, F') meets AB at point O so that $AO/OB = -(AF/FB)(AF'/F'B)$.

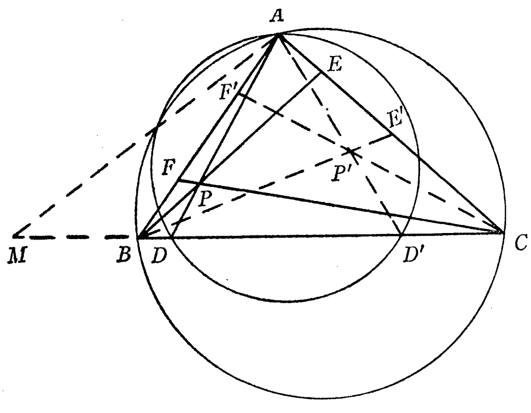

FIG. 2.

Suppose that P and P' are any two points in the plane of triangle ABC, with DEF and $D'E'F'$ their respective cevian triangles. Since Theorem 1 is true for any pair of circles (B, C) and (D, D'), we may allow each of these circles to pass through vertex A of the given triangle (Fig. 2). Circle (B, C) then becomes the circumcircle of triangle ABC. In a similar fashion allow circles (C, A) and (E, E') to pass through vertex B and circles (A, B) and (F, F') to pass through vertex C. Circles (C, A) and (A, B), like circle (B, C), coincide with the circumcircle of the triangle. The radical axis of the circumcircle and circle ADD' now meets BC at point M so that $BM/MC = -(BD/DC)(BD'/D'C)$. Similarly the radical axis of the circumcircle and circle BEE' meets CA at N so that $CN/NA = -(CE/EA)(CE'/E'A)$, while the radical axis of the circumcircle and circle CFF' meets AB at O so that $AO/OB = -(AF/FB)(AF'/F'B)$. We then have

$$\frac{BM}{MC}\cdot\frac{CN}{NA}\cdot\frac{AO}{OB} = -\left(\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}\right)\left(\frac{BD'}{D'C}\cdot\frac{CE'}{E'A}\cdot\frac{AF'}{F'B}\right) = -1,$$

since each of the products $(BD/DC)(CE/EA)(AF/FB)$ and $(BD'/D'C)\cdot(CE'/E'A)(AF'/F'B)$ is equal to $+1$ by Ceva's theorem. So $(BM/MC)\cdot(CN/NA)(AO/OB)=-1$ and the converse of the theorem of Menelaus shows that the points M, N, O are collinear.

THEOREM 2. *Let P and P' be any two points in the plane of triangle ABC, with DEF and $D'E'F'$ their respective cevian triangles. Let the radical axis of the circumcircle and circle ADD' meet BC at point M, that of the circumcircle and circle BEE' meet CA at point N, and that of the circumcircle and circle CFF' meet AB at point O. Then $BM/MC=-(BD/DC)\cdot(BD'/D'C)$, $CN/NA=-(CE/EA)\cdot(CE'/E'A)$, $AO/OB=-(AF/FB)\cdot(AF'/F'B)$ and points M, N, O are collinear.*

Let point P' of Theorem 2 be the orthocenter of triangle ABC. It is then known that

$$\frac{BD'}{D'C}=\frac{a^2+c^2-b^2}{a^2+b^2-c^2},\quad \frac{CE'}{E'A}=\frac{a^2+b^2-c^2}{b^2+c^2-a^2},\quad \frac{AF'}{F'B}=\frac{b^2+c^2-a^2}{a^2+c^2-b^2},$$

where a, b, c represent sides BC, CA, AB of the given triangle. Angle $AD'D$ is now a right angle and AD becomes a diameter of circle ADD'. This special case of Theorem 2 follows:

THEOREM 2A. *Let P be any point in the plane of triangle ABC, with DEF its cevian triangle. The three radical axes of the circumcircle and each of the three circles having for diameters rays AD, BE, CF meet sides BC, CA, AB at respective points M, N, O. Points M, N, O are collinear and*

$$\frac{BM}{MC}=-\left(\frac{a^2+c^2-b^2}{a^2+b^2-c^2}\right)\frac{BD}{DC},\quad \frac{CN}{NA}=-\left(\frac{a^2+b^2-c^2}{b^2+c^2-a^2}\right)\frac{CE}{EA},$$

$$\frac{AO}{OB}=-\left(\frac{b^2+c^2-a^2}{a^2+c^2-b^2}\right)\frac{AF}{FB}.$$

Theorem 2A has been derived by others through the use of alternate methods [1]. The author is inclined to believe, however, that ratio values for BM/MC, CN/NA, AO/OB of the preceding theorem have not been previously recorded.

Let a conic in the plane of triangle ABC cut side BC at points D and D', side CA at points E and E', and side AB at points F and F'. From Carnot's theorem it is known that $(BD/DC)\cdot(BD'/D'C)\cdot(CE/EA)\cdot(CE'/E'A)\cdot(AF/FB)\cdot(AF'/F'B)=1$, irrespective of whether rays AD, BE, CF and AD', BE', CF' are concurrent or not. Theorem 2 may now be further generalized in this manner:

THEOREM 2B. *Any conic in the plane of triangle ABC meets side BC at points D and D', side CA at points E and E', and side AB at points F and F'. The three radical axes of the circumcircle and circles ADD', BEE', CFF' meet respective sides BC, CA, AB at three collinear points.*

Let P again be any point in the plane of triangle ABC, with DEF its cevian triangle. Allow any straight line to meet sides BC, CA, AB at respective points M, N, O. Construct circles ADM, BEN, CFO and let the radical axis of the circumcircle and circle ADM meet BC at D', that of the circumcircle and circle BEN meet CA at E', and that of the circumcircle and circle CFO meet AB at F'. Theorem 1 yields $BD'/D'C = -(BD/DC)(BM/MC)$, $CE'/E'A = -(CE/EA)\cdot(CN/NA)$, $AF'/F'B = -(AF/FB)(AO/OB)$. Hence

$$\frac{BD'}{D'C}\cdot\frac{CE'}{E'A}\cdot\frac{AF'}{F'B} = -\left(\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}\right)\left(\frac{BM}{MC}\cdot\frac{CN}{NA}\cdot\frac{AO}{OB}\right).$$

Ceva's theorem tells us that $(BD/DC)(CE/EA)(AF/FB) = 1$, while the theorem of Menelaus yields $(BM/MC)(CN/NA)(AO/OB) = -1$. Therefore $(BD'/D'C)\cdot(CE'/E'A)(AF'/F'B) = 1$ and radical axes AD', BE', CF' must be concurrent by the converse of Ceva's theorem.

P', the point of concurrency of the three radical axes, has equal power with respect to the four circles ABC, ADM, BEN, CFO and is therefore their radical center. Consequently, ratio values are now known for the radical center of the four circles.

THEOREM 3. *P is any point in the plane of triangle ABC, with DEF its cevian triangle. Points M, N, O are any trio of collinear points on sides BC, CA, AB. The three radical axes of the circumcircle and circles ADM, BEN, CFO are concurrent at P', the radical center of the four circles. Ratio values for point P' are $BD'/D'C = -(BD/DC)\cdot(BM/MC)$, $CE'/E'A = -(CE/EA)\cdot(CN/NA)$, $AF'/F'B = -(AF/FB)\cdot(AO/OB)$.*

Let point P be the orthocenter so that rays AM, BN, CO become diameters of circles ADM, BEN, CFO. Theorem 3 may then be expressed in this fashion:

THEOREM 3A. *Let points M, N, O be any triad of collinear points on sides BC, CA, AB of triangle ABC. The three radical axes of the circumcircle and the three circles having for diameters rays AM, BN, CO meet sides BC, CA, AB at respective points D', E', F'. Rays AD', BE', CF' are concurrent at point P', the radical center of the four circles. Ratio values for point P' are*

$$\frac{BD'}{D'C} = -\left(\frac{a^2+c^2-b^2}{a^2+b^2-c^2}\right)\frac{BM}{MC}, \quad \frac{CE'}{E'A} = -\left(\frac{a^2+b^2-c^2}{b^2+c^2-a^2}\right)\frac{CN}{NA},$$

$$\frac{AF'}{F'B} = -\left(\frac{b^2+c^2-a^2}{a^2+c^2-b^2}\right)\frac{AO}{OB}.$$

Suppose that P and P' are now a pair of isogonal conjugates in triangle ABC, with DEF and $D'E'F'$ their respective cevian triangles. Any straight line through P' meets sides BC, CA, AB at points M, N, O respectively. Let the radical axis of the circumcircle and circle ADM meet BC at point D'', that of the circumcircle and circle BEN meet CA at point E'', and that of the circumcircle and circle CFO meet AB at point F''. By Theorem 3 rays AD'', BE'', CF'' are concurrent at a point P'' which is the radical center of circles ABC, ADM, BEN, CFO.

Since line MNO passes through P', it is known that $(AF'/F'B)\cdot(BO/OA) + (AE'/E'C)\cdot(CN/NA) = 1$ [2]. This equation yields

$$\frac{CN}{NA} = \frac{CE'}{E'A} - \frac{CE'}{E'A}\cdot\frac{AF'}{F'B}\cdot\frac{BO}{OA} = \frac{CE'}{E'A} - \frac{CD'}{D'B}\cdot\frac{BO}{OA}.$$

Here $CD'/D'B$ replaces $(CE'/E'A)(AF'/F'B)$ since Ceva's equation $(BD'/D'C) \cdot(CE'/E'A)(AF'/F'B)=1$ shows that the two quantities are equivalent. As points P and P' are isogonal conjugates, it is further known that $(BD/DC) \cdot(BD'/D'C)=(c^2/b^2)$ and $(CE/EA)\cdot(CE'/E'A)=(a^2/c^2)$ or $(CD'/D'B)=(b^2/c^2) \cdot(BD/DC)$ and $(CE'/E'A)=(a^2/c^2)\cdot(AE/EC)$. If these values be substituted in the expression just obtained for CN/NA, we have

$$\frac{CN}{NA} = \frac{CE'}{E'A} - \frac{CD'}{D'B}\cdot\frac{BO}{OA} = \frac{a^2}{c^2}\cdot\frac{AE}{EC} - \frac{b^2}{c^2}\cdot\frac{BD}{DC}\cdot\frac{BO}{OA}. \tag{1}$$

Finally, the use of Theorem 1 gives

$$CE''/E''A = -(CE/EA)(CN/NA) \tag{2}$$

and

$$BF''/F''A = -(BF/FA)(BO/OA). \tag{3}$$

Let us now evaluate the expression

$$\frac{b^2}{a^2}\cdot\frac{BF''}{F''A} + \frac{c^2}{a^2}\cdot\frac{CE''}{E''A}.$$

Using the values of $BF''/F''A$ and $CE''/E''A$ given in (3) and (2), we find that

$$\frac{b^2}{a^2}\cdot\frac{BF''}{F''A} + \frac{c^2}{a^2}\cdot\frac{CE''}{E''A} = -\frac{b^2}{a^2}\cdot\frac{BF}{FA}\cdot\frac{BO}{OA} - \frac{c^2}{a^2}\cdot\frac{CE}{EA}\cdot\frac{CN}{NA}.$$

If the value of CN/NA as given in (1) be substituted in the right member of this latter equation, and if $(BD/DC)(CE/EA)$ be replaced by BF/FA in the final term of the expression thus obtained, we eventually have

$$\frac{b^2}{a^2}\cdot\frac{BF''}{F''A} + \frac{c^2}{a^2}\cdot\frac{CE''}{E''A} = -1.$$

This means that P'', the point of intersection of radical axes AD'', BE'', CF'', must lie on circumcircle ABC [3]. P'' then has zero power with respect to circumcircle ABC. Being the radical center of the four circles, it must likewise have zero power with respect to the other three circles. Consequently, the four circles pass through point P'' and this interesting result follows:

THEOREM 3B. *P and P' are a pair of isogonal conjugates in triangle ABC, with DEF the cevian triangle of point P. A straight line rotating about P' meets sides BC, CA, AB at respective points M, N, O. Circles ABC, ADM, BEN, CFO have a point in common.*

It is well known that the four circumcircles of the four triangles formed by four lines in a plane have a point in common. Theorem 3B shows how to determine four other triangles that possess the same property.

Points P and P' of the preceding theorem coincide when P is the incenter or one of the three excenters of the given triangle. Theorem 3B then holds for line MNO rotating about either of these four points.

Let line MNO of Theorem 3B pass through both P' and Q, the isogonal and isotomic conjugates respectively of point P. If GHI is the cevian triangle of point Q, the ratio values for points P' and Q are

$$\frac{BD'}{D'C}=\frac{c^2}{b^2}\cdot\frac{CD}{DB},\qquad \frac{CE'}{E'A}=\frac{a^2}{c^2}\cdot\frac{AE}{EC},\qquad \frac{AF'}{F'B}=\frac{b^2}{a^2}\cdot\frac{BF}{FA}$$

and

$$\frac{BG}{GC}=\frac{CD}{DB},\qquad \frac{CH}{HA}=\frac{AE}{EC},\qquad \frac{AI}{IB}=\frac{BF}{FA}\,.$$

Line MNO passes through point P' and we may write $(AF'/F'B)\cdot(BO/OA)+(AE'/E'C)\cdot(CN/NA)=1$ or (see [2]).

$$\frac{b^2}{a^2}\cdot\frac{BF}{FA}\cdot\frac{BO}{OA}+\frac{c^2}{a^2}\cdot\frac{CE}{EA}\cdot\frac{CN}{NA}=1. \tag{4}$$

Since line MNO contains point Q, we also have $(AI/IB)\cdot(BO/OA)+(AH/HC)\cdot(CN/NA)=1$ or

$$\frac{BF}{FA}\cdot\frac{BO}{OA}+\frac{CE}{EA}\cdot\frac{CN}{NA}=1. \tag{5}$$

The simultaneous solution of equations (4) and (5) yields

$$\frac{CN}{NA}=\left(\frac{a^2-b^2}{c^2-b^2}\right)\frac{AE}{EC}\quad\text{and}\quad\frac{AO}{OB}=\left(\frac{b^2-c^2}{a^2-c^2}\right)\frac{BF}{FA}\,.$$

As CN/NA and AO/OB are now known, the theorem of Menelaus shows that

$$\frac{BM}{MC}=\left(\frac{c^2-a^2}{b^2-a^2}\right)\frac{CD}{DB}\,.$$

For point P'', the radical center of circles ABC, ADM, BEN, CFO, we have $(BD''/D''C)=-(BD/DC)\cdot(BM/MC)$. Replacing BM/MC by the value obtained at the end of the preceding paragraph, this becomes $(BD''/D''C)=(c^2-a^2)/(a^2-b^2)$. In similar fashion $(CE''/E''A)=(a^2-b^2)/(b^2-c^2)$ and $(AF''/F''B)=(b^2-c^2)/(c^2-a^2)$ so that point P'' becomes the Steiner point of triangle ABC [4, Theorem 7].

THEOREM 3C. *Let P be any point in the plane of triangle ABC, with DEF its cevian triangle. A straight line passing through the isogonal and isotomic conjugates of point P meets sides BC, CA, AB at respective points M, N, O. Circles ABC, ADM, BEN, CFO pass through the Steiner point of triangle ABC.*

What a multitude of circles through the Steiner point as P assumes all positions in the plane of triangle ABC!

Let the sides of triangle ABC be met by two straight lines MNO and $M'N'O'$. Circumcircle ABC and circles AMM', BNN', COO' may now be constructed. By paralleling the steps taken in the proof of Theorems 2 and 3, the reader may easily obtain this result:

THEOREM 4. *Two straight lines in the plane of triangle ABC meet sides BC, CA, AB at respective points M, N, O and M', N', O'. The three radical axes of the circumcircle and circles AMM', BNN', COO' meet sides BC, CA, AB at three collinear points.*

Suppose that D and D' are a pair of points that harmonically separate vertices B and C so that $BD'/D'C = -BD/DC$. Construct circles ABC and ADD' and let their radical axis meet BC at D''. By Theorem 1, $BD''/D''C = -(BD/DC) \cdot (BD'/D'C) = (BD/DC)^2$. A way is thus found to construct the square of given ratio BD/DC.

THEOREM 5. *Let D and D' be a pair of points that harmonically separate vertices B and C of triangle ABC. The radical axis of circles ABC and ADD' meets BC at point D'' so that $BD''/D''C = (BD/DC)^2$.*

We could now construct circle $AD''D'$ and let the radical axis of this circle and circumcircle ABC meet BC at point D'''. Then $(BD'''/D'''C) = -(BD''/D''C) \cdot (BD'/D'C) = (BD/DC)^3$ and a ratio representing the cube of BD/DC would be determined. It is clear that the process could be continued indefinitely so that $(BD/DC)^n$ could be constructed where n represents any positive integer.

The reader interested in the geometry of the triangle may wish to record variations of the preceding theorems as well as some of the converse facts. The author has obtained a number of interesting results by allowing circles (B, C), (C, A), (A, B) to pass through a fixed point Q, with circles (D, D'), (E, E'), (F, F') constructed in various ways. However, these results must await future consideration.

As usual, the equations and results recorded in this discussion are always true provided the segments involved are treated as directed quantities. When D lies between B and C, ratio BD/DC is considered positive. If D lies on BC extended, then ratio BD/DC must be considered negative. Similar comments apply to the other ratios involved in the results given.

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AN EXTENSION OF THE BUTTERFLY PROBLEM

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The well-known butterfly problem states that if P is the midpoint of an arbitrary chord AB of a circle, then $CP=PD$ where GI and HJ are any other two arbitrary chords passing through P. (See Figure 1) The name of the problem arises from the resemblance of the figure $G\ P\ I\ J\ P\ H\ G$ to a butterfly. Also, $EP=PF$. An elementary synthetic proof of this result is somewhat involved. A projective proof using poles and polars is given by Eves [1].

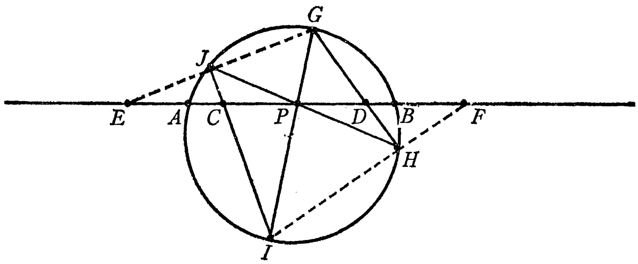

FIG. 1.

A variation of this problem states that if AB is a diameter of a circle with center O, and if $MO=ON$, then $CO=OD$ where JH and GI are arbitrary chords passing through M and N, respectively. (See Figure 2.) Also, $EO=OF$. This result is due to Cantab [2].

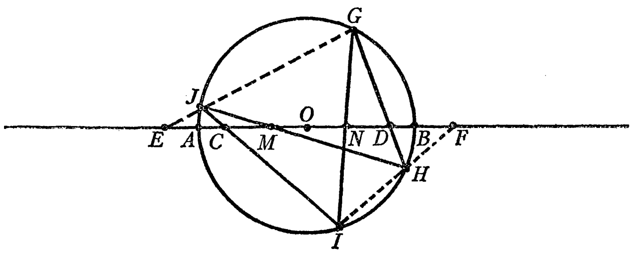

FIG. 2.

A generalization of both of these results is given by the following:

If AB is an arbitrary chord of a circle with midpoint P, and if $MP=PN$, then $CP=PD$ where JH and GI are arbitrary chords passing through M and N, respectively. Also, $EP=PF$. (See Figure 3.)

Our proof will be an analytic one using complex numbers. Our coordinate system is chosen such that the origin coincides with the center O of the circle (of unit radius) and such that $OP=ai$. Also, denote the complex representation for the four vertices G, H, I, J, by z_1, z_2, z_3, z_4, respectively, and let $PN=n$. Since

$$z_3 - n - ai = k(z_1 - n - ai) \qquad (k \text{ real}),$$

$$\cos\theta_3 - n = k(\cos\theta_1 - n),$$

$$\sin\theta_3 - a = k(\sin\theta_1 - a), \qquad (z_r = e^{i\theta_r}).$$

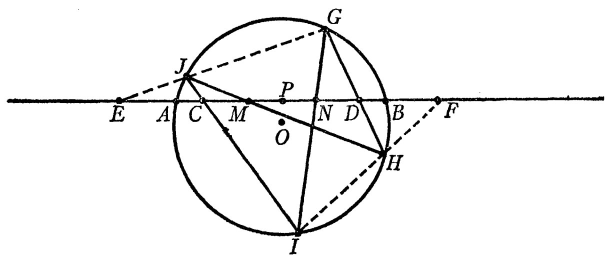

FIG. 3

Eliminating k:

$$n = \frac{\sin(\theta_1 - \theta_3) + a(\cos\theta_1 - \cos\theta_3)}{\sin\theta_1 - \sin\theta_3},$$

or equivalently,

$$(1) \qquad PN = \frac{1 + T_1T_3}{1 - T_1T_3} - a\,\frac{T_1 + T_3}{1 - T_1T_3}, \qquad (T_r = \tan\theta_r/2).$$

Similarly,

$$(2) \qquad PM = \frac{1 + T_2T_4}{1 - T_2T_4} - a\,\frac{T_2 + T_4}{1 - T_2T_4},$$

$$(3) \qquad PD = \frac{1 + T_1T_2}{1 - T_1T_2} - a\,\frac{T_1 + T_2}{1 - T_1T_2},$$

$$(4) \qquad PC = \frac{1 + T_3T_4}{1 - T_3T_4} - a\,\frac{T_3 + T_4}{1 - T_3T_4}.$$

By some elementary algebra,

$$\begin{aligned}(PM + PN)&(1 - T_1T_3)(1 - T_2T_4)\\ &= 2(1 - T_1T_2T_3T_4) - a[(T_1 + T_3)(1 - T_2T_4) + (T_2 + T_4)(1 - T_1T_3)],\\ &= 2(1 - T_1T_2T_3T_4) - a[(T_1 + T_2)(1 - T_3T_4) + (T_3 + T_4)(1 - T_1T_2)],\\ &= (PC + PD)(1 - T_1T_2)(1 - T_3T_4).\end{aligned}$$

Consequently,

$$PM + PN = 0 \Leftrightarrow PC + PD = 0 \text{ and, similarly, } \Leftrightarrow PE + PF = 0,$$

for the cases when none of the other factors vanish i.e., $1 - T_1T_4 \neq 0$, etc. When $1 - T_1T_4 = 0$, etc., we get obvious symmetric cases, e.g., $\theta_1 + \theta_4 = 180°$.

Since midpoints are invariant under an affine transformation, the previous results also hold for ellipses. (It would be of interest to give a projective proof here.) It seems natural to conjecture at this point, that for the class of convex curves, the previous properties characterize ellipses. One could also conjecture

that we need only have the "butterfly property" for one given point on a given chord to obtain a characterization of ellipses.

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1. Howard Eves, A survey of geometry, Allyn and Bacon, Boston, 1963, p. 171.
2. Problem No. 1549, Mathematical Questions from The Educational Times, 2 (1863) 67.

A "LATTICE POINT" PROOF OF THE INFINITUDE OF PRIMES

PAUL R. CHERNOFF, Princeton University

Suppose that there are only k primes, $p_1, p_2, \cdots, p_k$. Then every positive integer has a unique representation of the form $p_1^{e_1}p_2^{e_2}\cdots p_k^{e_k}$, where the e_i are nonnegative integers. Hence if N is any positive integer there are exactly N k-tuples of nonnegative integers $(e_1, e_2, \cdots, e_k)$ satisfying the inequality

$$p_1^{e_1}p_2^{e_2}\cdots p_k^{e_k} \leqq N$$

or, equivalently,

$$e_1 \log p_1 + e_2 \log p_2 + \cdots + e_k \log p_k \leqq \log N.$$

N, the number of such k-tuples, is thus the number of lattice points contained in the k-dimensional simplex

$$x_1 \log p_1 + x_2 \log p_2 + \cdots + x_k \log p_k \leqq \log N, \qquad x_i \geqq 0,$$

and is therefore asymptotic, as $N\to\infty$, to its volume, which is $(\log N)^k/(k! \log p_1 \cdots \log p_k)$. That is, we have

$$N \sim \text{constant}\cdot(\log N)^k,$$

which is certainly false. The number of primes is therefore infinite.

A NOTE ON EULER'S ϕ-FUNCTION

SADANAND VERMA, University of Windsor

The three theorems on Euler's ϕ-function—$\phi(n)=n\prod_{p|n}(1-1/p)$, $\sum_{d|n}\phi(d)=n$ and $\phi(m\cdot n)=\phi(m)\cdot\phi(n)$, for relatively prime m and n—are such that any one of them can be proved independently and the remaining two can be deduced thereafter. But the deduction of

$$\phi(n) = n\prod_{p|n}\left(1-\frac{1}{p}\right) \qquad \text{from } \sum_{d|n}\phi(d) = n$$

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The three theorems on Euler's ϕ-function—$\phi(n) = n\prod_{p|n}(1-1/p)$, $\sum_{d|n}\phi(d) = n$ and $\phi(m\cdot n) = \phi(m)\cdot\phi(n)$, for relatively prime m and n—are such that any one of them can be proved independently and the remaining two can be deduced thereafter. But the deduction of

$$\phi(n) = n\prod_{p|n}\left(1 - \frac{1}{p}\right) \qquad \text{from } \sum_{d|n}\phi(d) = n$$

requires a separate notion, that is of "Möbius function" (see for example, pp. 113–116 of [4] starting from the second proof). In fact, the proofs of many theorems in Number Theory depend rather on notions than on notations.

An English mathematician of the eighteenth century, Waring, in his book "Meditationes Algebraicae" (1770) reports a very interesting property of primes —more commonly known as "Wilson's Theorem"—communicated to him by an amateur in science, a certain Wilson. Confessing his inability to prove this theorem, Waring adds that the proof must be very difficult because there is no notation to designate primes only. Of this, Gauss observes that proofs of such theorems must depend rather on notions than on notations.

Nevertheless notations are not of less importance. For, each and every mathematical notion is expressed much more conveniently through proper notations, in so simple a form. Now, is it that the notion of "Möbius function" is indispensible for deducing

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right) \qquad \text{from } \sum_{d|n} \phi(d) = n?$$

Is it necessary to establish the Möbius inversion formula for this problem? Are the notion, notation, and properties of the divisors "d" of a number "n" not sufficient to obtain the canonical representation of $\phi(n)$? Let us wait and watch indeed as to what these questions mean. The following simple proof was devised by me about thirteen years ago and since then I have spent the intervening time, without success, in the hope of finding at least a similar proof in the literature. Finally, last year after writing to Professor Davenport of Cambridge (U.K.), I have come to the conclusion that such a simple proof does not exist in the literature.

Let $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_s^{\alpha_s}$; and as for any positive integer n,

$$\sum_{d|n} \phi(d) = n,$$

we have for the numbers of the type n/q_j, where q_j runs through all products of the distinct prime factors of n taken j at a time $(1 \leqq j \leqq s)$, the identity

$$\sum_{d|n} \phi(d) - \sum_{q_1} \sum_{d|n/q_1} \phi(d) + \cdots + (-1)^j \sum_{q_j} \sum_{d|n/q_j} \phi(d) \cdots + (-1)^s \sum_{d|n/q_s} \phi(d)$$
$$= n - \sum_{q_1} \frac{n}{q_1} + \cdots + (-1)^j \sum_{q_j} \frac{n}{q_j} \cdots + (-1)^s \frac{n}{q_s};$$

i.e., $\sum = n \prod_{p|n} (1-1/p)$, where $\sum$ stands for the first member of the identity and we write either of the equivalent forms $\sum_{i=1}^{s} n/p_i$ or $\sum_{q_1} n/q_1$ for the whole series $n/p_1 + n/p_2 + \cdots + n/p_s$ and so on for the other terms in the second member of the identity. In what follows, the term $\sum_{q_j} \sum_{d|n/q_j} \phi(d)$ of $\sum$ (the first member of the identity) will be designated by simply $\sum_{q_j}$ in short for each $j (1 \leqq j \leqq s)$.

Let us observe the types of terms that occur in the successive terms $\sum_{d|n} \phi(d)$, $\sum_{q_1}, \cdots, \sum_{q}$ of $\sum$ and the possible cancellation amongst them. First we note that any term $\phi(d_j)$ that occurs in $\sum_{q_j}$ arises out of a divisor, d_j, of a number of the form n/q_j and such divisors are divisors of the number n as well. Thus the types of terms that occur anywhere in $\sum$ involve divisors of n only. Choose an arbitrary divisor $d=p_1^{\beta_1}\cdot p_2^{\beta_2}\cdots p_s^{\beta_s}$ $(0\leqq\beta_i\leqq\alpha_i,\ i=1, 2, \cdots, s)$ of n other than $p_1^{\alpha_1}\cdot p_2^{\alpha_2}\cdots p_s^{\alpha_s}$ (i.e., the number n itself). Such a divisor must divide at least one of the numbers n/p_i $(i=1, 2, \cdots, s)$. Suppose it divides exactly μ of these numbers. Now follow this divisor through the terms in $\sum$ to note the exact number of times, with proper sign, it precipitates the term $\phi(d)$ in each of the terms $\sum_{d|n}\phi(d)$ $\sum_{q_1}, \cdots, \sum_{q_s}$ of $\sum$. Clearly this divisor, d, furnishes the term $\phi(d)$ once in $\sum_{d|n}\phi(d)$ and exactly$-\mu$ times in $-\sum_{q_1}$. Let P_k run through all products, taken k at a time $(1\leqq k\leqq\mu)$, of only those μ of the primes p_i $(i=1, 2, \cdots, s)$ and no other which occur in the denominators of the above μ of the numbers n/p_i $(i=1, 2, \cdots, s)$ that are divisible by the chosen divisor d. Obviously, this divisor d, in fact, divides all the numbers of the form n/P_k.

Therefore every such divisor of n, which gives rise to a term occurring μ times in $\sum_{q_1}$, must give rise to the same term occurring

$$\text{once in } \sum_{d|n}\phi(d),$$

$$-\binom{\mu}{1}\text{ times in } -\sum_{q_1},$$

$$+\binom{\mu}{2}\text{ times in } +\sum_{q_2},$$

$$\cdots\cdots\cdots\cdots$$

$$(-1)^\mu\binom{\mu}{\mu}\text{ times in } (-1)^\mu\sum_{q\mu},$$

where the

$$\binom{\mu}{k}=\frac{\mu(\mu-1)\cdots(\mu-k+1)}{k!}$$

are the binomial coefficients $(1\leqq k\leqq\mu)$. Thus the number of times such terms occur in $\sum$ (the first member of the identity) is

$$1-\binom{\mu}{1}+\binom{\mu}{2}-\cdots+(-1)^k\binom{\mu}{k}\cdots+(-1)^\mu\binom{\mu}{\mu}=(1-1)^\mu=0.$$

In addition to all such divisors of n there is one and only one left, which, being the number n itself, gives rise to the only term $\phi(n)$ occurring nowhere in $\sum$ except once in $\sum_{d|n}\phi(d)$. Hence the value of $\sum$, that is the first member of the identity, is simply $\phi(n)$. Incidentally, what has been proved is that

$$\phi(n)=n\prod_{p|n}\left(1-\frac{1}{p}\right)$$

without using the notion of "Möbius function."

This proof is really very simple when presented. But this is the beauty of Number Theory. As once Gauss wrote in one of his letters to a talented and learned lady, Sophie Germain: "Les charmes enchanteurs de cette science sublime ne se décèlent dans toute leur beauté qu'à ceux qui ont le courage de l'approfondir."

Summer Research Institute, The Canadian Mathematical Congress, Vancouver, 1964.

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2. L. E. Dickson, History of the theory of numbers, vol. 1, 2, and 3, Chelsea, New York, 1952.
3. Oystein Ore, Number theory and its history, McGraw-Hill, New York, 1948.
4. J. V. Uspensky and M. A. Heaslet, Elementary number theory, McGraw-Hill, New York, 1939.
5. I. M. Vinogradov, An introduction to the theory of numbers. Transl. by Helen Popova. Pergamon Press, New York, 1955.

ANSWERS

A364. If AC and BD intersect at I, then G, H, and I are collinear by Pappus' Theorem. Any line through I bisects the parallelogram.

A365. $(70+75+85+90)-300$ or 20%. In general,

$$P = \sum_{i=1}^{n} p_i - 100(n-1).$$

A366. Let x, y, z be the roots of $s^3+a_1s^2+a_2s+a_3=0$. Then

$$a_1 = 3, \qquad \sum x^2 = \left(\sum x\right)^2 - 2\sum xy \quad \text{and} \quad a_2 = 11/4.$$

Next

$$\sum x^3 + a_1\sum x^2 + a_2\sum x + 3a_3 = 0 \quad \text{and} \quad a_3 = -3/4.$$

The roots of the cubic are 1/2, 2/2, and 3/2.

A367. Suppose otherwise. If the world population is n, this implies that for each of the numbers $0, 1, 2, \cdots, n-1$, one person has that number of friends. But if one person is a friend of everyone else, no one is friendless. Thus the 0 and $n-1$ are incompatible.

A368. The equation $ba=b$ implies that a is the identity. The group has order 7, is therefore cyclic and generated by b. The given information yields $b^2=d$, $b^3=bd=c$, and $b^4=bc=f$. Then $ag=g$ implies that $bg\neq g$, thus $b^5=bf=g$ and $b^6=e$. The table follows from the laws of exponents.

(Quickies on page 252)

ON A SUBSTITUTION MADE IN SOLVING RECIPROCAL EQUATIONS

ARNOLD SINGER, Institute of Naval Studies (A division of the Center for Naval Analyses of the Franklin Institute), Cambridge, Mass.

The standard procedure employed in solving reciprocal equations requires the substitution

$$y = x + 1/x. \tag{1}$$

One is then required to write x^2+1/x^2, $x^3+1/x^3, \cdots$, as polynomials in y. Many texts give the relationships up to x^4+1/x^4 or so, and some give the recurrence relation

$$x^{n+1} + 1/x^{n+1} = (x^n + 1/x^n)y - (x^{n-1} + 1/x^{n-1}).$$

This note derives the expression for x^n+1/x^n as a polynomial in y for general n.

From (1) we obtain $x^2-xy+1=0$, or, $x=\frac{1}{2}y\pm\frac{1}{2}\sqrt{(y^2-4)}$. Therefore,

$$\begin{aligned} x^n + x^{-n} &= 2^{-n}[y + (y^2 - 4)^{1/2}]^n + 2^{-n}[y - (y^2 - 4)^{1/2}]^n \\ &= 2^{-n} \sum_0^\infty \binom{n}{k} [1 + (-1)^k] y^{n-k}(y^2 - 4)^{k/2} \\ &= 2^{-n+1} \sum_0^\infty \binom{n}{2m} y^{n-2m}(y^2 - 4)^m \\ &= 2^{-n+1} \sum_0^\infty \binom{n}{2m} y^{n-2m} \sum_j \binom{m}{j} (-4)^j y^{2m-2j} \\ &= 2^{-n+1} \sum_j (-4)^j y^{n-2j} \sum_m \binom{n}{2m}\binom{m}{j}. \end{aligned}$$

But

$$\sum_m \binom{n}{2m}\binom{m}{j} = 2^{n-2j-1} \frac{n}{n-j}\binom{n-j}{j}.$$

(See, e.g., E. Netto, *Kombinatorik*, Chelsea, equation 33, page 253.) Thus,

$$\begin{aligned} x^n + x^{-n} &= \sum_j (-1)^j \frac{n}{n-j}\binom{n-j}{j} y^{n-2j} \\ &= y^n - \frac{n}{n-1}\binom{n-1}{1} y^{n-2} + \frac{n}{n-2}\binom{n-2}{2} y^{n-4} - \cdots. \end{aligned}$$

SOLVING MAZE PUZZLES

SIDNEY KRAVITZ, Dover, New Jersey

Several years ago a puzzle contest was sponsored for advertising purposes. Each contestant was required to solve a series of increasingly difficult maze puzzles of the type described below. Following in the footsteps of Euler's "Seven Bridges of Königsberg" [1] a method is presented for solving these puzzles.

A simple example of the type of puzzle referred to is shown in Figure 1. Here we see several areas (marked by small circles and identified as $a, b, c, \cdots, h, i$), with each area given a numerical value (not shown). The contestant is required to trace a route between the starting point (marked S) and the final point (F) using only the paths shown (as dashed lines) and passing through each area no more than once. Numerical credit is given only for those areas which the route passes through. The contestant who achieves the highest numerical sum wins.

Let us begin the discussion with the thought that if we found a route which passes through every area then we would certainly get the maximum possible score. It would therefore be profitable to know whether such a route is possible or not. We will assume that such a route is possible and will use this assumption to lead either to the correct route or routes, or to a logical contradiction, thus proving that an all inclusive route is impossible. The maze puzzle in Figure 1 will be used to demonstrate the simpler aspects of the method.

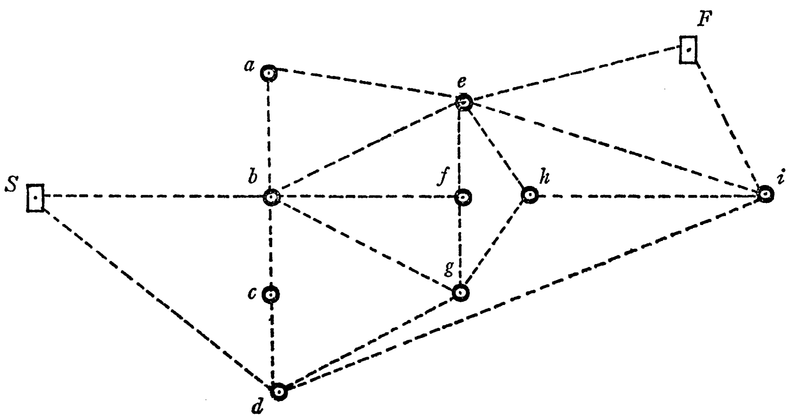

FIG. 1.

In order for any route to pass through an area only once it must utilize two (and only two) paths, one leading into the area and one away from it. We observe that areas a and c each have only two paths leading to it and, consequently, if an all inclusive route exists, it must utilize these available paths. Therefore the route must use paths ab and ae to enter or leave area a, and must use paths cd and bc to enter or leave area c. The new situation is shown in Figure 2. Here solid lines indicate those paths which the route must take and dashed lines indicate those paths which are still available.

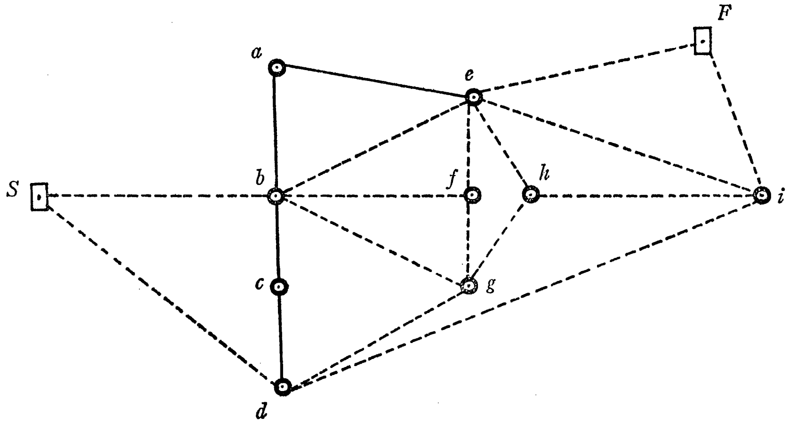

FIG. 2.

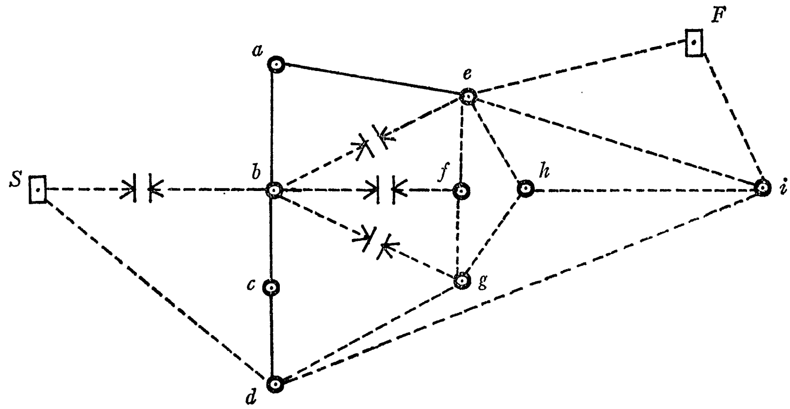

FIG. 3.

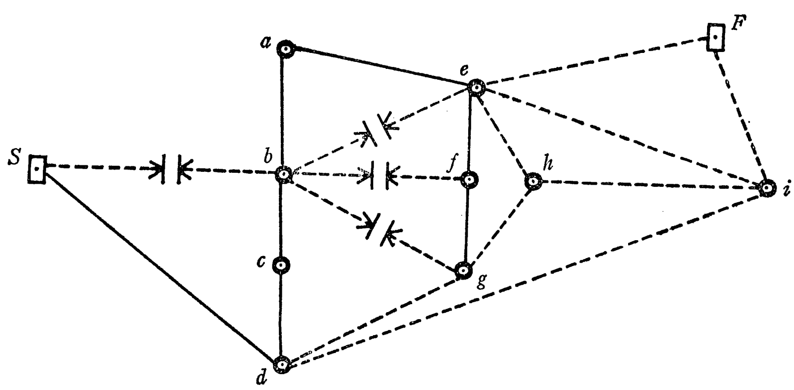

FIG. 4.

Figure 2 shows that we have already designated a route through area *b* namely, by using the paths *bc* and *ab*. Since the route must not pass through area *b* a second time, it follows that we must never utilize any of the other paths leading into or away from *b* i.e., paths *Sb*, *be*, *bf*, and *bg* must be made unavailable for further use. Figure 3 shows the new situation with the symbol -->||<-- used to designate a path which must not be used.

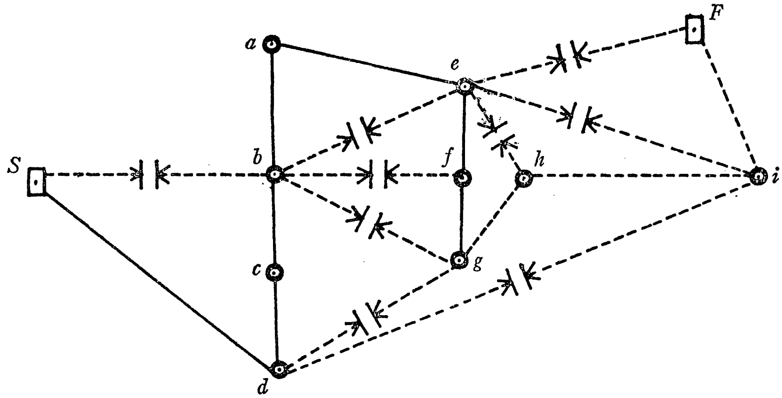

Fig. 5.

Figure 3 shows that if path *bf* is not to be used then area *f* has only two available paths leading to and from it, namely, paths *ef* and *fg*. Likewise if path *Sb* is not available, then we must utilize path *Sd* if we ever expect to leave the starting area *S*. We indicate these paths as solid lines in Figure 4. We now find that the route passes through *d* so that paths *dg* and *di* must be made unavailable. Likewise the route passes through *e* making *eF*, *ei*, and *eh* unavailable. See Figure 5. The only paths now open are *gh*, *hi*, and *iF*. The final solution is shown in Figure 6.

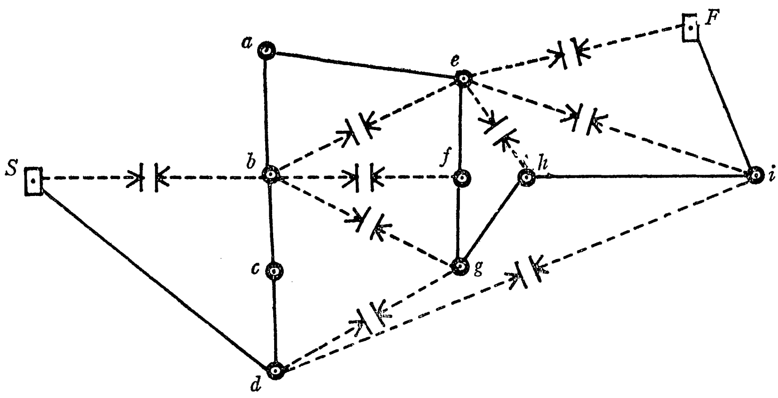

Fig. 6.

Several additional principles are demonstrated in the examples which follow. In Figure 7 we see that the route must go from *c* to *j* to *k* to *F*, and that the path *ik* must be considered unavailable. We next draw an imaginary closed loop (as shown in a dotted line). There are only two available paths into this imaginary loop, namely *de* and *ci*. If we are not to isolate all areas inside the loop we must use both of these paths. We are then able to complete the puzzle without recourse to trial and error.

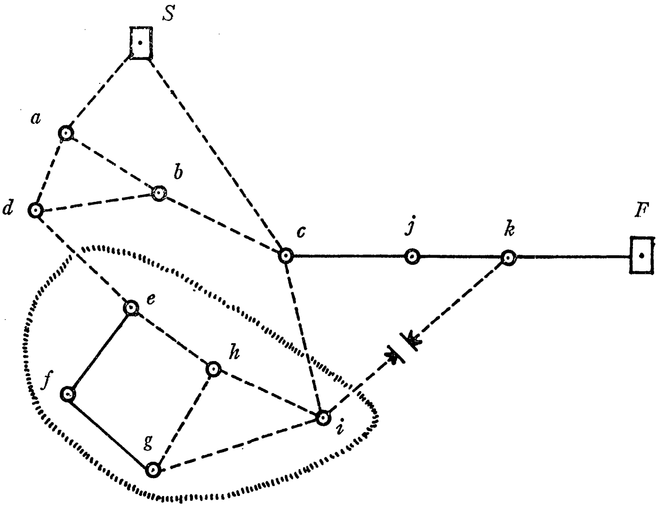

FIG. 7.

Consider the maze puzzle in Figure 8. The route from *S* to *F* must use the paths *Sa*, *dh*, *hi*, *bc*, *cf*, and *jF*. However, *di* is unavailable since it would form a closed polygon of solid lines with *dh* and *hi*. We find that three solid lines lead into the imaginary loop shown, namely *hi*, *cf*, and *Fj*. By one of these three paths we go into the loop, by another we go out, and then in again. It is obvious that we must utilize the one remaining available path, *bi*, to get out of the loop.

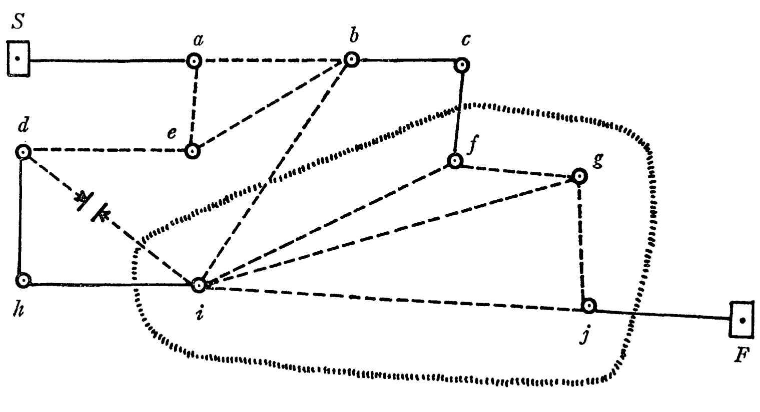

FIG. 8.

In general if any loop does not enclose the starting or final area then the route must lead in and out of the loop for a total of an even number of times. However if one and only one of the terminal points is within the loop this total must be an odd number.

Not all maze puzzles of this type have only one all inclusive solution. Some have several solutions and others cannot possibly include all areas. However with the methods outlined here the reader will be able in a matter of minutes, to find the highest possible score for puzzles which appear to be complicated. Figure 9 is presented as an exercise for the reader.

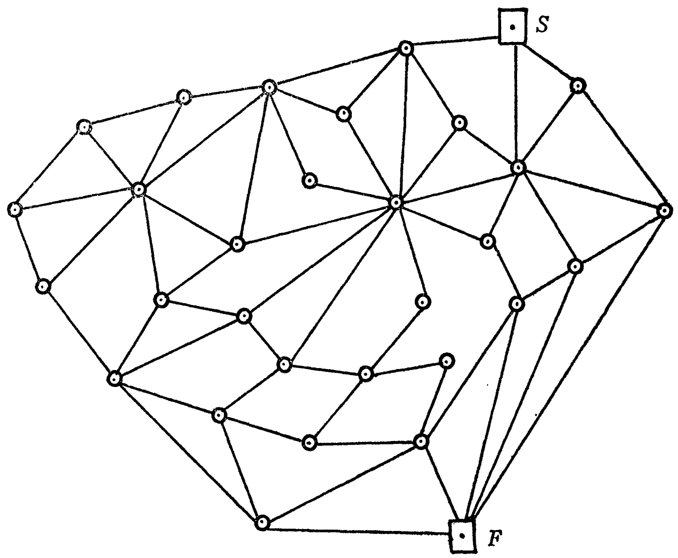

FIG. 9.

References

1. Leonhard Euler, The seven bridges of Königsberg in vol. I, The World of Mathematics, Simon and Schuster, New York, 1956, p. 573.

2. W. T. Tutte, A non-Hamiltonian graph, Canad. Math. Bull., 3 (1960) 1–5.

A PROOF OF FEUERBACH'S THEOREM

S. VENKATRAMAIAH, Tenali, India

Let S be the circum-centre, I the in-centre, and N the nine-points centre of $\triangle ABC$. Take S as the origin of vectors. Let **a**, **b**, **c** be the position vectors of A, B, C respectively. Then $(a\mathbf{a}+b\mathbf{b}+c\mathbf{c})/(a+b+c)$ and $\frac{1}{2}(\mathbf{a}+\mathbf{b}+\mathbf{c})$ are the position vectors of I and N.

In general if any loop does not enclose the starting or final area then the route must lead in and out of the loop for a total of an even number of times. However if one and only one of the terminal points is within the loop this total must be an odd number.

Not all maze puzzles of this type have only one all inclusive solution. Some have several solutions and others cannot possibly include all areas. However with the methods outlined here the reader will be able in a matter of minutes, to find the highest possible score for puzzles which appear to be complicated. Figure 9 is presented as an exercise for the reader.

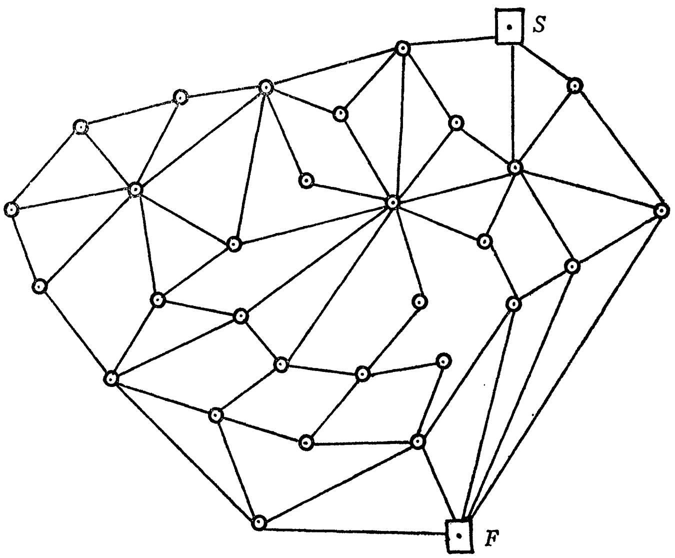

FIG. 9.

References

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Let S be the circum-centre, I the in-centre, and N the nine-points centre of $\triangle ABC$. Take S as the origin of vectors. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be the position vectors of A, B, C respectively. Then $(a\mathbf{a}+b\mathbf{b}+c\mathbf{c})/(a+b+c)$ and $\frac{1}{2}(\mathbf{a}+\mathbf{b}+\mathbf{c})$ are the position vectors of I and N.

$$\mathbf{IN} = \tfrac{1}{2}(\mathbf{a}+\mathbf{b}+\mathbf{c}) - (a\mathbf{a}+b\mathbf{b}+c\mathbf{c})/(a+b+c)$$
$$= (\tfrac{1}{2s})\,[(s-a)\mathbf{a}+(s-b)\mathbf{b}+(s-c)\mathbf{c}]$$
$$IN^2 = \tfrac{1}{4}(R^2/s^2)\left[\sum (s-a)^2 + 2\sum (s-a)\cdot(s-b)\,\text{Cos}\,2C\right]$$
$$(1)\qquad = \tfrac{1}{4}(R^2/s^2)\left[s^2 - 4\sum (s-a)\cdot(s-b)\,\text{Sin}^2\,C\right]$$
$$= \tfrac{1}{4}R^2 - R^2\frac{(4\,\text{Sin}\,\tfrac{1}{2}A\cdot\text{Sin}\,\tfrac{1}{2}B\cdot\,\text{Sin}\,\tfrac{1}{2}C)}{(4\,\text{Cos}\,\tfrac{1}{2}A\cdot\text{Cos}\,\tfrac{1}{2}B\cdot\text{Cos}\,\tfrac{1}{2}C)}\sum 2\,\text{Sin}\,C\cdot\text{Sin}^2\,\tfrac{1}{2}C.$$

Now

$$(2)\qquad \sum 2\,\text{Sin}\,C\cdot\text{Sin}^2\,\tfrac{1}{2}C = \sum \text{Sin}\,C(1-\text{Cos}\,C) = \sum \text{Sin}\,C - \sum \tfrac{1}{2}\,\text{Sin}\,2C$$
$$= (4\,\text{Cos}\,\tfrac{1}{2}A\cdot\,\cos\,\tfrac{1}{2}B\cdot\,\text{Cos}\,\tfrac{1}{2}C)(1 - 4\,\text{Sin}\,\tfrac{1}{2}A\cdot\text{Sin}\,\tfrac{1}{2}B\cdot\text{Sin}\,\tfrac{1}{2}C).$$

By (1) and (2) and using $r=4R\text{Sin}\,\tfrac{1}{2}A\cdot\text{Sin}\,\tfrac{1}{2}B\cdot\text{Sin}\,\tfrac{1}{2}C$ we get

$$NI^2 = \tfrac{1}{4}R^2 - Rr(1 - 4\,\text{Sin}\,\tfrac{1}{2}A\cdot\text{Sin}\,\tfrac{1}{2}B\cdot\text{Sin}\,\tfrac{1}{2}C) = \tfrac{1}{4}R^2 - Rr + r^2 = (r - \tfrac{1}{2}R)^2.$$

Hence the in-circle touches the nine-points circle.

Let I_1 be the ex-centre opposite to A. The position vector of I_1 is

$$(-a\mathbf{a}+b\mathbf{b}+c\mathbf{c})/2(s-a)$$
$$\mathbf{I_1N} = [s\mathbf{a} - (s-c)\mathbf{b} - (s-b)\mathbf{c}]/2(s-a).$$

Proceeding as in the previous case we get that

$$I_1N^2 = \tfrac{1}{4}R^2 + R^2$$
$$\cdot\left[\frac{s(s-c)}{(s-a)(s-a)}\,\text{Sin}^2\,C - \frac{(s-c)(s-b)}{(s-a)(s-a)}\,\text{Sin}^2\,A + \frac{s(s-b)}{(s-a)(s-a)}\,\text{Sin}^2\,B\right]$$
$$(3)\qquad = \tfrac{1}{4}R^2 + R^2\,\frac{(4\,\text{Sin}\,\tfrac{1}{2}A\cdot\text{Cos}\,\tfrac{1}{2}B\cdot\text{Cos}\,\tfrac{1}{2}C)}{(4\,\text{Cos}\tfrac{1}{2}A\cdot\text{Sin}\,\tfrac{1}{2}B\cdot\text{Sin}\,\tfrac{1}{2}C)}$$
$$\cdot\,[2\,\text{Cos}^2\,\tfrac{1}{2}C\cdot\text{Sin}\,C - 2\,\text{Sin}^2\,\tfrac{1}{2}A\cdot\text{Sin}\,A + 2\,\text{Cos}^2\,\tfrac{1}{2}B\cdot\text{Sin}\,B].$$

Now

$$2\,\text{Cos}^2\,\tfrac{1}{2}C\cdot\text{Sin}\,C - 2\,\text{Sin}^2\,\tfrac{1}{2}A\cdot\text{Sin}\,A + 2\,\text{Cos}^2\,\tfrac{1}{2}B\cdot\text{Sin}\,B$$
$$= (1+\text{Cos}\,C)\,\text{Sin}\,C - (1-\text{Cos}\,A)\,\text{Sin}\,A + (1+\text{Cos}\,B)\,\text{Sin}\,B$$
$$(4)\qquad = (\text{Sin}\,C - \text{Sin}\,A + \text{Sin}\,B) + \tfrac{1}{2}\sum \text{Sin}\,2A$$
$$= (4\,\text{Cos}\,\tfrac{1}{2}A\cdot\text{Sin}\,\tfrac{1}{2}B\cdot\text{Sin}\,\tfrac{1}{2}C)(1 + 4\,\text{Sin}\,\tfrac{1}{2}A\cdot\text{Cos}\,\tfrac{1}{2}B\cdot\text{Cos}\,\tfrac{1}{2}C).$$

By (3) and (4), using the formula $r_1=4R\text{Sin}\,\tfrac{1}{2}A\cdot\text{Cos}\,\tfrac{1}{2}B\cdot\text{Cos}\,\tfrac{1}{2}\text{C}$, we get that

$$I_1N^2 = \tfrac{1}{4}R^2 + Rr_1 + r_1^2 = (\tfrac{1}{2}R + r_1)^2.$$

Hence the Nine-Points circle touches the ex-circle opposite to A. Similarly we can show that the Nine-Points circle touches the ex-circles opposite to B and C.

DISTANCE FROM LINE TO POINT

DOUGLAS H. MOORE, California State Polytechnic College, Pomona

The discussions by Mott [1] and Brown [2] of the treatment of sensed distance from line to point in two dimensional analytical geometry, prompts me to offer the following presentation for which I am indebted to many teachers, books, and snowed students.

This presentation does not assume vector algebra is available—but even if it were I would offer the following point of view to a class for its additional instructional value. I will restrict the formal discussion to the case of a line, l, parallel to neither axis:

$$l: \ Ax + By + C = 0, \qquad AB \neq 0$$

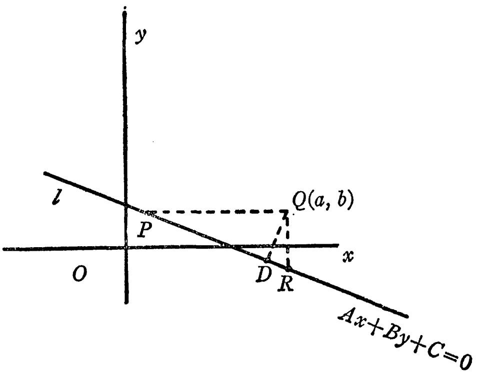
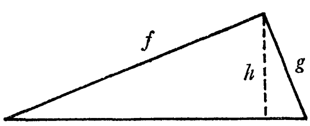


FIG. 1.

Let $Q(a, b)$ be any point in the plane. Let lines through Q parallel to the x and y axes intersect l at P and R respectively. Let D be the foot of the perpendicular from Q to l.

The discussion will be in two parts; in Part I, we use the formula:

$$h = \frac{fg}{\sqrt{(f^2 + g^2)}} \tag{1}$$



for the height of a right triangle, as shown, to obtain the unsensed distance DQ in Fig. 1. In Part II, we use the continuity of the expression:

$$Ax + By + C \tag{2}$$

in the two variables x and y (with no formal treatment of "continuity") to show that l divides the plane into three parts in which (2) is positive, negative, and zero respectively, and we learn to tell, in three different ways by a glance at (2), whether (2) is positive or negative at a chosen point not on l.

Part I.

$$Ax + By + C = 0.$$

Subtract $Aa+Bb$ (where $Q=(a, b)$) from each side and transpose to obtain:

$$A(a - x) + B(b - y) = Aa + Bb + C.$$

$$\left.\begin{array}{ll}\text{Setting } y = b: & PQ = |a - x| = \left|\dfrac{Aa + Bb + C}{A}\right| \\ \text{Setting } x = a: & RQ = |b - y| = \left|\dfrac{Aa + Bb + C}{B}\right|\end{array}\right\}$$ Thus formulas for PQ and RQ in Fig. 1 are obtained.

Using (1):

$$DQ = \frac{(PQ)(RQ)}{\sqrt{(PQ)^2 + (RQ)^2}} = \frac{(Aa + Bb + C)^2/|AB|}{|Aa + Bb + C|\sqrt{\left(\dfrac{1}{A^2} + \dfrac{1}{B^2}\right)}} = \frac{|Aa + Bb + C|}{\sqrt{(A^2 + B^2)}}.$$

Part II. Two points in the plane are on the same side of l if the straight line segment joining them has no point in common with l. Two points in the plane are on opposite sides of l if the straight line segment joining them has an interior point in common with l.

THEOREM. *The expression* (2) *has the same sign at any two points on the same side of l, and has opposite signs at two points on opposite sides of l.*

Proof. From earlier work (2) is zero at each point of l, and l includes all points in the plane where (2) is zero.

Let M and N be two points on the same side of l. Then the line segment, MN, does not cross l, and so (2) is not zero at any point on MN. Therefore, due to the continuity of (2), it assigns values of the same sign to M and N.

But (2) assigns positive values to some points in the plane and negative values to others; e.g., let $y=0$ and give x a large positive value and then a large negative value, remembering that $A\neq 0$. With a little more thought we have: q.e.d.

The expressions:

(3) $\quad Ax + By + C \qquad$ (4) $\quad -Ax - By - C$

both vanish on l. That expression, (2) or (4), whose constant term is positive (in the case when $C\neq 0$) has a positive value at the origin, O, and so has a positive value at all points on the same side of l as the origin. This provides one way of determining what sign (2) has at a given point not on l, except when l passes through the origin.

In Fig. 1 suppose Q is "to the right of l"; i.e., the x-coordinate of Q exceeds that of P; then that expression, (2) or (4), whose x-coefficient is positive, has a larger value of Q than at P and so it has a *positive* value at Q, and at all points to the right of l. This provides a second way of determining the sign of (2).

In Fig. 1 suppose Q is "above l"; i.e., the y-coordinate of Q exceeds that of R; then that expression, (2) or (4), whose y-coefficient is positive has a larger value at Q than at R and so it has a *positive* value at Q, and at all points above l. This provides a third way of determining the sign of (2).

Finally reintroduce "DQ" as the *sensed* distance from l to $Q(a, b)$, given by one of the formulas:

$$(5) \qquad DQ = \frac{Aa + Bb + C}{\sqrt{(A^2 + B^2)}} \qquad\qquad (6) \qquad DQ = \frac{-Aa - Bb - C}{\sqrt{(A^2 + B^2)}}.$$

If we adopt the convention of choosing that equation, (5) or (6), in which the coefficient of a is positive, then DQ is positive when Q is to the right of l and negative to the left.

If we adopt the convention of choosing that equation, (5) or (6), in which the coefficient of b is positive, then DQ is positive when Q is above l and negative below.

If we adopt the convention of choosing that equation, (5) or (6), in which the "constant term" is positive, then DQ is positive when Q is on the same side of l as the origin and negative on the opposite side.

The facts about the sign of (2) may be presented as follows.

(a) $Ax + By + C$ has the same sign as A to the right of l

(b) $Ax + By + C$ has the same sign as B above l

(c) $Ax + By + C$ has the same sign as C (when $C \neq 0$) on the side of l where the origin is found.

If $A = 0$, (b) and (c) still hold.

For example. $3y-4$ is positive above the line $3y-4=0$ and is negative on the side of the origin.

If $B=0$, (a) and (c) still hold.

For example. $-3x+4$ is negative to the right of the line $-3x+4=0$ and is positive on the side of the origin.

If $C=0$, (a) and (b) still hold.

For example. $-3x+4y$ is negative to the right of the line $-3x+4y=0$ and is positive above the line.

Exercise. Generalize this approach to three dimensions.

References

1. Thomas Mott, The distance formula and conventions for sign, this MAGAZINE, 35 (1962) 39–42.

2. T. A. Brown, The distance of a point from a line, this MAGAZINE, 37 (1964) 157–159.

NOTE CONCERNING TWO CONSTRUCTION PROBLEMS IN GEOMETRY

KAIDY TAN, Fukien Normal College, Foochow, China

In this article we will consider two well-known construction problems and point out certain erroneous results which have been published concerning these problems.

PROBLEM 1. Construct a triangle, given its altitudes.

Kutuzov [1] gives the solution as follows (abbreviated here):

Noting that

$$ah_a = bh_b = ch_c = 2S \tag{1}$$

(where S is the area of $\triangle ABC$; a, b, c are the sides opposite A, B, C respectively; and h_a, h_b, h_c are the altitudes on sides a, b, c, respectively), he concludes that

$$a:b:c = \frac{1}{h_a}:\frac{1}{h_b}:\frac{1}{h_c}\,. \tag{2}$$

From this he derives the construction which follows.

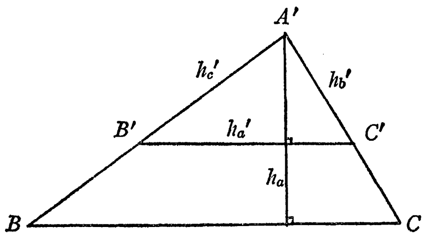


FIG. 1.

Construct a triangle with sides h_a, h_b, h_c and label the altitudes of this triangle h_a', h_b', h_c'. Clearly, $a:b:c = h_a':h_b':h_c'$. Now construct a triangle $A'B'C'$ with sides h_a', h_b', h_c'. Using homothetic figures, take A' as the homothetic center and the required triangle is easily found as shown in Figure 1.

In conclusion he points out that this problem is soluble if and only if

$$h_a + h_b > h_c. \tag{3}$$

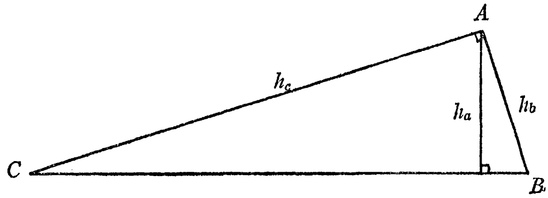


FIG. 2a.

But it is not necessary that the sum of two altitudes of a triangle be greater than the third altitude. On the contrary, consider $\triangle ABC$ (Fig. 2a) where $\measuredangle A = 90°$. Then, $h_a = h_c \sin C$, $h_b = h_c \tan C$, and therefore $h_a + h_b = h_c (\sin C + \tan C)$. If $0° < C < 27°58'$, then $\sin C + \tan C < 1$ (by tables). Hence if $0° < C < 27°58'$, $h_a + h_b < h_c$ but the triangle exists. Figure 2b illustrates a case of the oblique triangle where $h_a + h_b < h_c$.

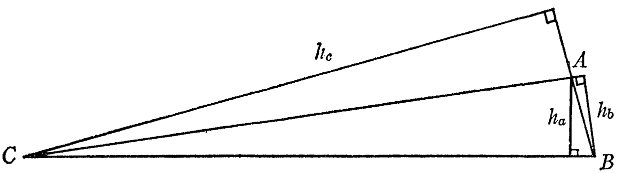


FIG. 2b.

We see that the above method, while it seems ingenious, is not general; it fails when $h_a + h_b \leqq h_c$.

A better method may be found in Court's work [2]. Prof. Court starts with (1) and divides each member by $h_a h_b$, obtaining

$$\frac{a}{h_b} = \frac{b}{h_a} = \frac{c}{m}, \tag{4}$$

where $m = (h_a h_b / h_c)$. m is readily constructed as the fourth proportional to the three given altitudes. By (4) the required triangle is similar to the triangle DEF, where $EF = h_b$, $FD = h_a$, and $DE = m$ (Fig. 3). On the altitude DK of this triangle lay off $DL = h_a$. The parallel through L to EF meets DE, DF in the vertices B, C of the required $\triangle ABC$, whose third vertex A coincides with D.

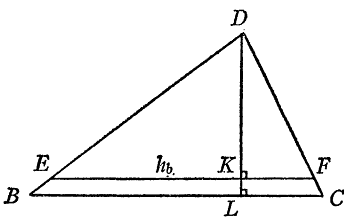


FIG. 3.

Finally, he points out that this problem is soluble if and only if $h_a + h_b > m > h_a - h_b$, or, replacing m by its value and dividing by $h_a h_b$:

$$\frac{1}{h_b} + \frac{1}{h_a} > \frac{1}{h_c} > \frac{1}{h_b} - \frac{1}{h_a}. \tag{5}$$

This solution is general, complete, and exact.

Now we shall give two alternate methods; perhaps they will stimulate the interest of the reader.

Solution 1. Utilizing the power of a point with respect to a circle.

We could easily solve our problem if we could construct a triangle similar to the required triangle, i.e., if we could find three line segments proportional to a, b, c. From (1) we may recall the theorem on the power of a point with respect to a circle. If we apply this theorem we may draw any circle Ω and take a suitable point O so that the three rays OP, OQ, OR drawn from O to the circle Ω are $OP=h_a$, $OQ=h_b$, $OR=h_c$ (Fig. 4). Let the secants OP, OQ, OR cut the circle again in the points P', Q', R' respectively. Let $OP'=a'$, $OQ'=b'$, and $OR'=c'$.

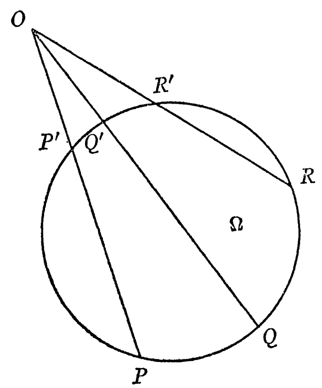


FIG. 4.

Then

(6) $$a'h_a = b'h_b = c'h_c$$

Dividing (6) by (1), we get

(7) $$\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c}.$$

That is, the triangle having sides a', b', c' is similar to the required triangle. By the method of similitude the required triangle can be constructed immediately as follows:

Construct triangle $A'B'C'$ (see Fig. 5) having sides a', b', c'. Draw $A'D \perp B'C'$, cutting off $A'D=h_a$. The parallel to $B'C'$ through D meets $A'B'$, $A'C'$ in the vertices B, C of the required triangle ABC whose third vertex A coincides with A'. The proof is simple and we omit it here.

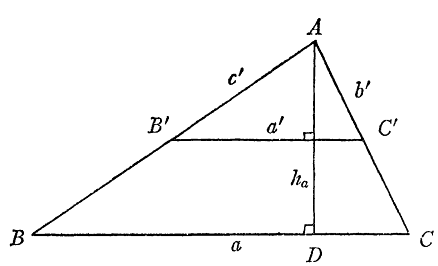


FIG. 5.

From (2) we have

$$\frac{a}{\dfrac{1}{h_a}} = \frac{b}{\dfrac{1}{h_b}} = \frac{c}{\dfrac{1}{h_c}} = \frac{a+b}{\dfrac{1}{h_a}+\dfrac{1}{h_b}} = \frac{a-b}{\dfrac{1}{h_a}-\dfrac{1}{h_b}}.$$

Since in any triangle $a+b>c>a-b$, this reduces to

$$\frac{1}{h_a}+\frac{1}{h_b} > \frac{1}{h_c} > \frac{1}{h_a}-\frac{1}{h_b}.$$

This result agrees with (5).

Solution 2. Utilizing the properties of a circle and similar figures.

Let ABC be the required triangle with altitudes $AD=h_a$, $BE=h_b$, and $CF=h_c$. Since $\measuredangle ADB$ and $\measuredangle AEB$ are right angles, D and E lie on the circle with diameter AB (see Fig. 6). On this circle draw the chord AP so that $\measuredangle DAP=\measuredangle CAB$. Then $DP=BE=h_b$. Now, since $\triangle APD\sim\triangle ABC$, $(AD/AP)=(AC/AB)$. Also, since $\triangle ACF\sim\triangle ABE$, $(AC/AB)=(CF/BE)$. Hence, $(AD/AP)=(CF/BE)$. That is, AP is the fourth proportional to the three altitudes CF, BE, and AD. Accordingly, $\triangle APD$ can be constructed and the required triangle ABC can be constructed as follows:

Construct the triangle APD having as its sides h_a, h_b, and (h_ah_b/h_c). Draw the circumcircle of triangle APD and draw the chord DB perpendicular to AD. Next draw the chord $BE=DP$, E and D being on the same side of AB. Let AE and BD meet at C; triangle ABC will be the required triangle. The proof is simple and we omit it here.

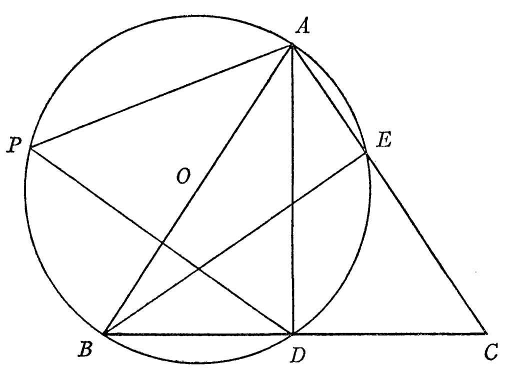


FIG. 6.

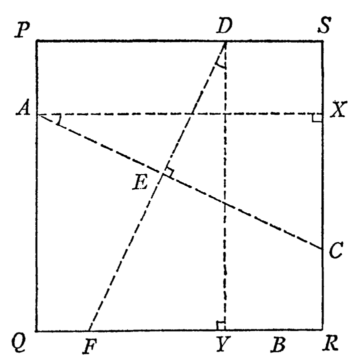


FIG. 7.

PROBLEM 2. Construct a square so that each side, or the side produced, shall pass through a given point.

The solution below appears in the works of both Court and Davis [3] and is due to M. Biandsutter (MATHESIS, 1881, p. 8).

Let $PQRS$ be the required square whose sides PQ, QR, RS, SP pass, respectively, through the points A, B, C, D (Fig. 7). From the point D draw the perpendicular DE to AC meeting the line QR at F. Draw AX perpendicular to

RS and DY perpendicular to QR. Clearly $AX=DY$ and $\sphericalangle XAC=\sphericalangle YDF$ (sides mutually perpendicular). Therefore right triangle XAC is congruent to right triangle YDF and $DF=AC$. Hence F is easily determined and the line BF is easily determined. This observation offers the key to the following solution:

Draw AC joining two of the given points. From a third given point D draw the perpendicular to AC. On the perpendicular cut off $DF=AC$. Draw BF. Then from A and C draw the perpendiculars to BF. The parallel to BF through D completes the construction. Again, the proof is simple and we omit it.

An examination of the construction shows that any two of the given points may be joined and to this line a perpendicular may be drawn from either of the remaining two points. On this perpendicular can be measured, in either direction from the third point, a distance equal to the distance between the first two points. A solution results in every case, even if some or all of the given points are collinear.

F. G. M. Exercices [4] states that this problem has four solutions. This is incomplete. Davis' work states that the total number of solutions is 24, and that in general these solutions are all distinct. This conclusion is a mistake. The correct number is given in the excellent work of Court, together with the figure of the complete solutions. This problem has in general six different solutions.

Now we give an alternate solution of this problem for comparison with the above method.

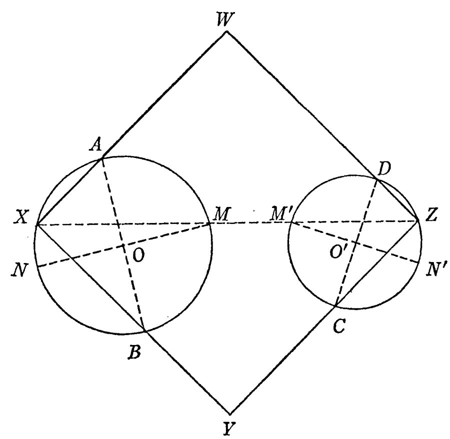


FIG. 8.

Let $XYZW$ be a required square on each side of which lies one of the four given points A, B, C, D (Fig. 8). Now, since $\sphericalangle AXB$ is a right angle, X must lie on the circle O with diameter AB. Similarly, Z must lie on the circle O' with diameter CD. Since the diagonal XZ bisects $\sphericalangle X$ and $\sphericalangle Z$ it must pass through the midpoints M and M' of the semicircles AMB and $CM'D$. From this we derive the following construction:

Construct the circles O and O' with diameters AB and CD respectively. In each circle draw the perpendicular diameter, $MN \perp AB$ and $M'N' \perp CD$. Draw the straight line MM' cutting the two circles in X and Z. Draw the lines XA, XB, ZC, ZD and we get one of the required squares.

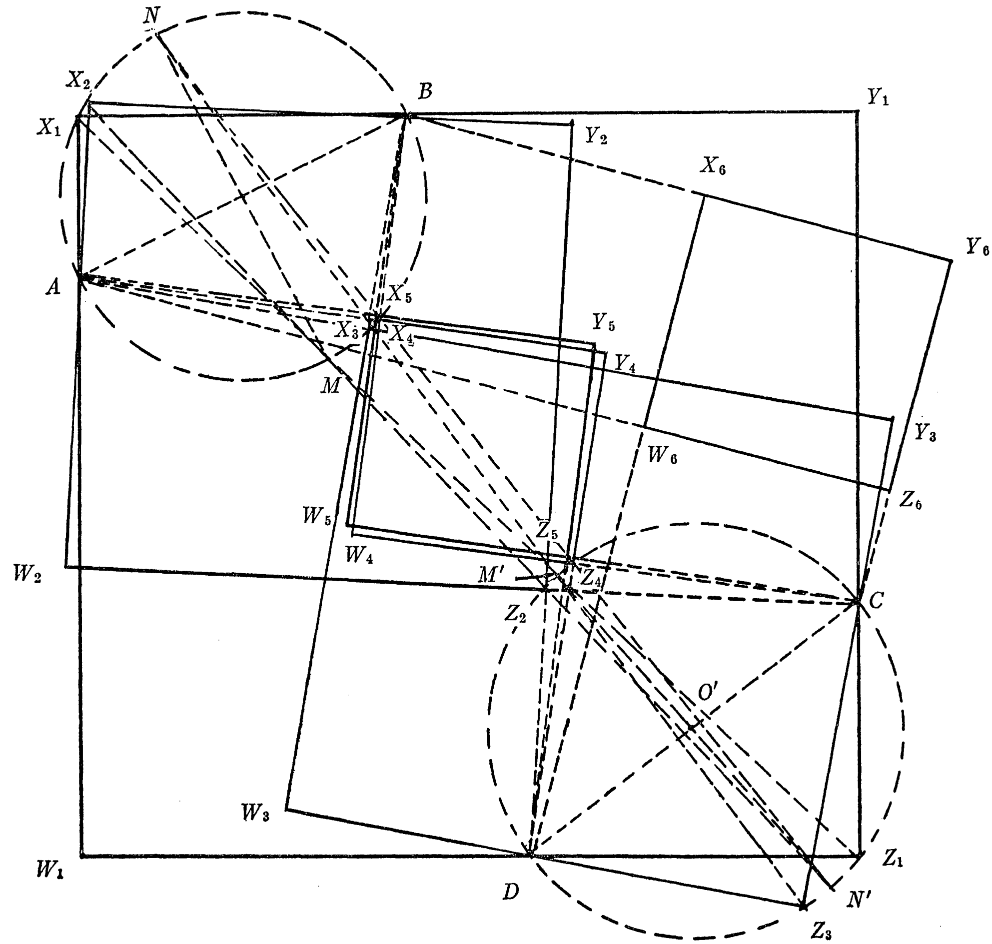


FIG. 9.

If we draw NN', MN', or $M'N$ instead of MM', we get some of the other required squares. Under such circumstances we get the four squares 1, 2, 3, 4 (Fig. 9 where 1 denotes the square $X_1Y_1Z_1W_1$, etc.). If the circles are constructed with AD and BC or with AC and BD as diameters instead of AB and CD, we get the sets of four squares $\{5, 6, 1, 2\}$ and $\{3, 4, 5, 6\}$. Hence, using all possible combinations of diameters we get three sets of four squares but they coincide by twos. There are generally six distinct solutions. This agrees with the conclusion in Court's work. Collinearity of points has no influence upon the number of solutions. This is also pointed out by Prof. Davis.

However, if the four points are the vertices of a rhombus there are four distinct solutions only! For in this case if we construct the two circles with a pair of opposite sides as diameters, they are equal circles and tangent externally at the center of the rhombus. Hence two of the squares degenerate into a point (the point of contact of the two circles, i.e., the center of the rhombus) and disappear. Unfortunately this special case is not mentioned in any book that I know of.

If the four points are the vertices of a square, then the two circles constructed on a pair of opposite sides as diameters are tangent externally at the point P, the center of the square. In this case XZ may revolve about P and in any position will determine a required square. Under these circumstances we get an infinite number of required squares and the problem becomes indefinite. This is also shown in Court's work from another point of view.

Prepared for publication by Ronald R. DeLaite, University of Maine.

References

1. A. B. Kutuzov, Geometria (Russian edition), 2nd ed., 1955, Moscow.
2. N. A. Court, College geometry, 2nd ed., Barnes and Noble, New York, 1952.
3. A. B. Davis, Modern college geometry, Cambridge, Mass., 1954.
4. F. G. M., Exercices de Géométrie, 6th ed., Paris, 1920.

A THEOREM ANALOGOUS TO MORLEY'S THEOREM

HUSEYIN DEMIR, Middle East Technical University, Ankara, Turkey

In Morley's theorem [1] one starts with an arbitrary triangle ABC and by trisecting the angles A, B, C arrives at an equilateral triangle. In this paper we state a property, by which, starting with an equilateral triangle ABC and dividing the angles into three parts arbitrarily by positive angles α, β, γ such that $\alpha+\beta+\gamma=60°$, one arrives at a triangle $A'B'C'$ whose angles are 3α, 3β, 3γ.

THEOREM. *Let ABC be an equilateral triangle and let α, β, γ be any three positive angles such that $\alpha+\beta+\gamma=60°$. Proceeding clockwise, the angle A is divided into β, α, γ in that order; B is divided into γ, β, α in that order, and C is divided into α, γ, β in that order. Let A', B', and C' be points in the interior of the triangle such that*

$$\begin{array}{lll} \angle BAB' = \gamma, & \angle B'AC' = \alpha, & \angle C'AC = \beta, \\ \angle CBC' = \alpha, & \angle C'BA' = \beta, & \angle A'BA = \gamma, \\ \angle ACA' = \beta, & \angle A'CB' = \gamma, & \angle B'CB = \alpha. \end{array}$$

Then the triangle $A'B'C'$ so obtained has angles A', B', C' equal to 3α, 3β, 3γ respectively.

Proof. Letting $BC=CA=AB=1$, we express $a'=B'C'$, $b'=C'A'$, $c'=A'B'$ in terms of the angles α, β, γ (see figure). Applying the sine law to the triangles pair of opposite sides as diameters, they are equal circles and tangent externally at the center of the rhombus. Hence two of the squares degenerate into a point (the point of contact of the two circles, i.e., the center of the rhombus) and disappear. Unfortunately this special case is not mentioned in any book that I know of.

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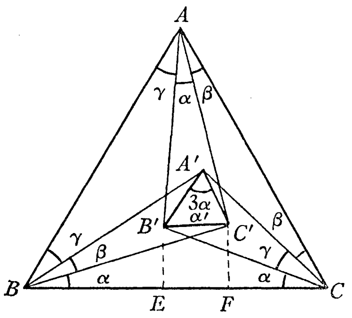

ABC' and $AB'C$, we have

$$(1)\qquad \begin{aligned} BC' &= \frac{\sin(60° - \beta)}{\sin(60° + \gamma)} = \frac{\cos(30° + \beta)}{\cos(30° - \gamma)} \\ CB' &= \frac{\sin(60° - \gamma)}{\sin(60° + \beta)} = \frac{\cos(30° + \gamma)}{\cos(30° - \beta)} \cdot \end{aligned}$$

Denoting the orthogonal projections of B', C' on the side BC by E, F respectively, we have

$$a'^2 = (FC' - EB')^2 + EF^2,$$

where the difference $FC' - EB'$ may be negative. Hence

$$(2)\qquad \begin{aligned} a'^2 &= (BC' \sin\alpha - CB' \sin\alpha)^2 + (BF + EC - BC)^2 \\ a'^2 &= [(BC' - CB') \sin\alpha]^2 + [(BC' + CB') \cos\alpha - 1]^2. \end{aligned}$$

Now, setting

$$(3)\qquad 2u = 2\cos(30° - \beta)\cos(30° - \gamma) = \cos\alpha + \cos(\beta - \gamma)$$

and using (1), we get

$$\begin{aligned} BC' \pm CB' &= \frac{\cos(30° + \beta)}{\cos(30° - \gamma)} \pm \frac{\cos(30° + \gamma)}{\cos(30° - \beta)} \\ &= \frac{1}{2u}[2\cos(30° + \beta)\cos(30° - \beta) \pm 2\cos(30° + \gamma)\cos(30° - \gamma)] \\ &= \frac{1}{2u}[(\cos 60° + \cos 2\beta) \pm (\cos 60° + \cos 2\gamma)] \\ &= \frac{1}{2u}[(\cos 60° \pm \cos 60°) + (\cos 2\beta \pm \cos 2\gamma)] \end{aligned}$$

and

$$(4)\qquad \begin{aligned} 2u(BC' + CB') &= 1 + \cos 2\beta + \cos 2\gamma \\ 2u(BC' - CB') &= \cos 2\beta - \cos 2\gamma. \end{aligned}$$

Substituting (4) in (2) and using (3), we obtain

$$\begin{aligned}
4u^2a'^2 &= [(\cos 2\beta - \cos 2\gamma)\sin\alpha]^2 + [\cos\alpha + \cos\alpha(\cos 2\beta + \cos 2\gamma) - 2u]^2 \\
&= [-2\cos(30^\circ + \alpha)\sin(\beta - \gamma)\sin\alpha]^2 \\
&\quad + [\cos\alpha + 2\cos\alpha\cos(60^\circ - \alpha)\cos(\beta-\gamma) - \cos\alpha - \cos(\beta-\gamma)]^2 \\
&= 4\cos^2(30^\circ+\alpha)\sin^2\alpha\cdot[1 - \cos^2(\beta-\gamma)] \\
&\quad + \cos^2(\beta-\gamma)\cdot[2\cos\alpha\cos(60^\circ-\alpha) - 1]^2 \\
&= 4\cos^2(30^\circ+\alpha)\sin^2\alpha \\
&\quad + \cos^2(\beta-\gamma)\cdot[(2\cos\alpha\cos(60^\circ-\alpha)-1)^2 - (2\cos(30^\circ+\alpha)\sin\alpha)^2] \\
&= 4\cos^2(30^\circ+\alpha)\sin^2\alpha \\
&\quad + \cos^2(\beta-\gamma)\cdot[2\cos\alpha\sin(30^\circ+\alpha) - 2\sin\alpha\cos(30^\circ+\alpha) - 1] \\
&\qquad\qquad \cdot[2\cos\alpha\sin(30^\circ+\alpha) + 2\sin\alpha\cos(30^\circ+\alpha) - 1] \\
&= 4\cos^2(30^\circ+\alpha)\sin^2\alpha \\
&\quad + \cos^2(\beta-\gamma)\cdot[2\sin 30^\circ - 1]\cdot[2\sin(30^\circ+2\alpha) - 1] \\
&= 4\cos^2(30^\circ+\alpha)\sin^2\alpha.
\end{aligned}$$

Since $\cos(30^\circ+\alpha)$ and $\sin\alpha$ are nonnegative and u is given by (3) we have

$$2ua' = 2\cos(30^\circ+\alpha)\sin\alpha, \qquad a' = \frac{\cos(30^\circ+\alpha)\sin\alpha}{\cos(30^\circ-\beta)\cos(30^\circ-\gamma)}\cdot$$

Setting $v=\cos(30^\circ-\alpha)\cos(30^\circ-\beta)\cos(30^\circ-\gamma)$ we have

$$\begin{aligned}
va' &= \cos(30^\circ+\alpha)\cos(30^\circ-\alpha)\sin\alpha \\
2va' &= (\cos 60^\circ + \cos 2\alpha)\sin\alpha \\
4va' &= (1 + 2\cos 2\alpha)\sin\alpha = \sin 3\alpha.
\end{aligned}$$

Similarly, it may be shown that

$$4vb' = \sin 3\beta, \qquad 4vc' = \sin 3\gamma.$$

Therefore, since $v\neq 0$,

$$a':b':c' = \sin 3\alpha : \sin 3\beta : \sin 3\gamma$$

which implies that

$$A' = 3\alpha, \qquad B' = 3\beta, \qquad C' = 3\gamma.$$

This completes the proof of the theorem.

Reference

1. H. S. M. Coxeter, Introduction to geometry, Wiley, New York, 1963, pp. 23–25.

ANOTHER INTERESTING PROPERTY OF TWO CONSECUTIVE NUMBERS

CHARLES T. SALKIND, Polytechnic Institute of Brooklyn

We establish herein that any integral power of the number $\sqrt{a}-\sqrt{(a-1)}$, a an integer, $a \geqq 2$, is expressible in the form $\sqrt{A}-\sqrt{(A-1)}$, A an integer, $A \geqq 2$.

An illustration or two will make the meaning and the power of the theorem clearer. The square of $\sqrt{2}-1$ can be written as $\sqrt{9}-\sqrt{8}$ and the cube of $\sqrt{2}-1$, as $\sqrt{50}-\sqrt{49}$, and, similarly, for higher powers. The square of $\sqrt{3}-\sqrt{2}$ is equal to $\sqrt{25}-\sqrt{24}$ and the cube of $\sqrt{3}-\sqrt{2}$ equals $\sqrt{243}-\sqrt{242}$, and, similarly, for higher powers.

Specifically, then, we are required to prove that $(\sqrt{a}-\sqrt{(a-1)})^n$, n, a integers, $a \geqq 2$, can be written as $\sqrt{A}-\sqrt{(A-1)}$, A an integer, $A \geqq 2$. The case where $a=1$ is trivial. The proof proceeds by induction on n.

The result is trivially true for $n=1$. For $n=2$, we have

$$(\sqrt{a} - \sqrt{(a-1)})^2 = (2a-1) - 2\sqrt{(a(a-1))} = \sqrt{(4a^2-4a+1)} - \sqrt{(4a^2-4a)},$$

and it is obvious that the two radicands differ by 1.

Assume, then, that

$$(\sqrt{a} - \sqrt{(a-1)})^{2k} = S - R\sqrt{(a(a-1))} = \sqrt{S^2} - \sqrt{(a(a-1)R^2)}$$

is of the form $\sqrt{A}-\sqrt{(A-1)}$; that is, $S^2-a(a-1)R^2=1$. Then

$$\begin{aligned}(\sqrt{a} - \sqrt{(a-1)})^{2k+2} &= (\sqrt{a} - \sqrt{(a-1)})^{2k}(\sqrt{a} - \sqrt{(a-1)})^2 \\ &= (S - R\sqrt{(a(a-1))})(2a-1) - 2\sqrt{(a(a-1))} \\ &= [S(2a-1) + 2Ra(a-1)] - \sqrt{(a(a-1))}[(2a-1)R + 2S].\end{aligned}$$

Let $S'=S(2a-1)+2Ra(a-1)$ and $R'=(2a-1)R+2S$. Then

$$\begin{aligned}S'^2-a(a-1)R'^2 &= S^2(4a^2-4a+1)+4RSa(a-1)(2a-1)+R^2(4a^4-8a^3+4a^2) \\ &\quad -S^2(4a^2-4a)-4RSa(a-1)(2a-1)-R^2(4a^4-8a^3+5a^2-a) \\ &= S^2-a(a-1)R^2.\end{aligned}$$

Since $S^2-a(a-1)R^2$ is, by assumption, equal to 1, $S'^2-a(a-1)R'^2$ equals to 1. Similarly, it can be shown that, if $(\sqrt{a}-\sqrt{(a-1)})^{2k-1}$ is in the form $V\sqrt{(a(a-1))}-U$ such that $a(a-1)V^2-U^2=1$, then, too, $(\sqrt{a}-\sqrt{(a-1)})^{2k+1}$ is of the form $\sqrt{(a(a-1))}\,V'-U'$ such that $a(a-1)V'^2-U'^2=1$.

This establishes the theorem for all integral values of n.

$N^2+21N+1$ AS A GENERATOR OF PRIMES

J. A. H. HUNTER, Toronto, Canada

So far as I know, there has been no mention of this polynomial as a generator of primes amongst the many that have received attention since Euler's discovery of his famous polynomial N^2-N+41. The function $N^2+21N+1$, however, does have properties which seem worthy of note.

If we allow the use of negative numbers, $N^2+21N+1$ generates 28 noncomposite integers (i.e., 27 primes and unity) with successive values $N=-10$ to $N=17$.

We use the notation $H(m)$ to mean the value of N, in $N^2+21N+1$, which generates the smallest product of m distinct and different prime factors [1] and $f\{H(m)\}$ to denote the value of $N^2+21N+1$ corresponding to $H(m)$.

Then we have:

m	$H(m)$	$f\{H(m)\}$
2	18	$19 \cdot 37$
3	208	$19 \cdot 23 \cdot 109$
4	1519	$19 \cdot 23 \cdot 53 \cdot 101$
5	17251	$19 \cdot 23 \cdot 47 \cdot 89 \cdot 163$

Suppose $N^2+21N+1=19^nk$, where k is some integer. Then, $2N=-21 \pm\sqrt{(437+4\cdot 19^nk)} = -21\pm\sqrt{\{19(23+4\cdot 19^{n-1}k)\}}$, which requires $23+4\cdot 19^{n-1}k =19t^2$, say. But this can have no integral solution for $n>1$. Hence $N^2+21N+1$ cannot have a square or higher power of 19 as a factor. The same applies to the square or a higher power of 23.

References

1. J. A. H. Hunter, Further results with N^2-N+A, this MAGAZINE, 36 (1963) 313–314.
2. Sidney Kravitz, Euler's prime generating polynomial, this MAGAZINE, 35 (1962) 152.

BOOK REVIEWS

EDITED BY DMITRI THORO, San Jose State College

Materials intended for review should be sent to: Dmitri Thoro, Department of Mathematics, San Jose State College, San Jose, California 95114.

A Programmed Introduction to Number Systems. By Irving Drooyan and Walter Hadel. Wiley, New York, 1964. ix+292 pp. Paperbound $3.95.

This brief exposition on number systems develops the basic concepts of sets and operations on sets, the system of natural numbers, the system of integers, and the system of rational numbers. It is appropriate reading for both preservice and in-service elementary teachers.

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For the pre-service elementary teacher *A Programmed Introduction to Number Systems* would serve very well as a supplementary text for a CUPM level I mathematics course. Many topics usually included in a level I course in arithmetic, however, are not covered in the program. Among them are numeration systems, elementary number theory, and an introduction to the real number system.

For the in-service teacher, the major advantage of this text over other texts of similar content is the feature of self instruction. Instruction is presented in the form of linear programming, a format which often results in dull, tedious reading, but the authors have avoided this fault of programmed materials by varying the presentation with appropriate remarks, summaries, and self evaluation tests. The book is very readable.

Another pedagogical feature is the method of presenting proofs. The task of teaching the nature of deductive proof to either pre-service or in-service teachers, as anyone who has taught a course of this type knows, is a particularly formidable one. A clear and repeated distinction is made between axioms and theorems, with the nature of an axiom clearly stated and re-emphasized throughout. Preceding the proof of each theorem, the strategy of the proof is sketched in a numerical example. Other applications of theorems and axioms to numerical exercises occur throughout.

The approach to number systems in the Drooyan and Hadel text is that of extending a previously postulated or defined set of numbers to create a new number system. That is, a given set of numbers is accepted as a subset of the new set of numbers, with some other elements included to form the new set. Thus the rational numbers are defined as follows: "The set of rational numbers contains the integers and such numbers as are necessary to insure that every pair of integers (a, b), $b \neq 0$ will have a quotient." The axioms of equality, order, and operations for the integers are accepted as axioms of the system of rationals invoking the "principle of permanence" and the existence of a multiplicative inverse is postulated. The usual definitions of equivalence, addition, and multiplication are then derived as theorems.

Whether one likes or dislikes this approach to establishing a number system will undoubtedly determine if *A Programmed Introduction to Number Systems* is an acceptable text.

For the unsophisticated reader, it seems that a scheme whereby "how to add" and "how to multiply" are proved rather than defined (which would be the alternate approach) makes a great deal of sense. Most students using a text of this type have come through an elementary and secondary school mathematics program knowing how to perform the operations on whole numbers, integers, and rational numbers. They would now like to know *why* the rules work.

MARILYN ZWENG, University of Iowa

Statistical Concepts, a Program for Self-Instruction. By Celeste McCollough and Loche Van Atte. McGraw-Hill, New York, 1963. xv+367 pp. $3.95.

This text, programmed in the linear (nonbranching) fashion, adopts a pace and a level which makes it suitable for use by a student whose mathematical background consists of only high school algebra. The lessons cover probability, hypothesis testing, linear regression, estimation, and some of the usual elementary applications of the binomial, normal, Chi-square, and t distributions. Most of the illustrative examples are drawn from the field of psychology—as might be expected since the authors are teachers of that subject and initially wrote the material for use in a psychology course. The emphasis is on the development of concepts and not on computational techniques. The programming seems to be well done so that the text can actually be used for self-instruction. There is also a teacher's manual available in case the book is used in a classroom. The book itself may be had in a more expensive hard-cover edition.

Unfortunately, there are quite a few logical or mathematical errors in some of the frames. For example, the following statement is made when X is a continuous variable: "If the usual .05 level of significance is chosen, we must be ready to reject the null hypothesis if the probability of getting $\overline{X}$ when $\mu=100$ is 0.05 or less." A bright student whose high school background included one of the new more careful treatments of algebra would find in the book many instances of sloppy usage.

In the elucidation of the concept of probability, the primitive intuitive notion is taken to be *expected frequency.* To this reviewer, this seems a poor choice from both the logical and pedagogical points of view. Conditional probabilities are never introduced. Binomial distributions are used in examples, but never identified as such.

In spite of these weaknesses, certain portions of the book might be helpful to a student who is having trouble with some of the standard topics in an elementary statistics course.

F. L. WOLF, Carleton College

Elementary Concepts of Modern Mathematics. By Flora Dinkines. Appleton-Century-Crofts, New York, 1964. x+457 pp., $6.50. Paperbound under three separate titles: *Elementary Theory of Sets*, $2.45. *Introduction to Mathematical Logic*, $1.45. *Abstract Mathematical Systems*, $1.45.

In *Elementary Concepts of Modern Mathematics*, Flora Dinkines has produced a very readable book in three parts; "Part I" developing the theory of sets, "Part II" concerned with an introduction to mathematical logic and "Part III" devoted to the discussion of abstract mathematical systems.

Included in the treatment of set theory besides the usual handling of set operations, set relations, Cartesian products, power sets, set diagrams (Venn diagrams are properly distinguished from Euler circles) and the like are leisurely treatments of Boolean algebras, successor sets, finite and infinite sets, counta-

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In her "Introduction to Mathematical Logic," Miss Dinkines develops the sentential calculus and the restricted predicate calculus in a fairly straight forward manner although I find several points where I feel she departs from usually accepted terminology without advantage and several points where explication is either lacking and needed, or unclear. The author restricts the use of the term 'implication' to those conditional statements which never assume a truth value of F. Such use of the term is peculiar, to say the least, since in most systems 'implication' and 'conditional' are used interchangeably; that is the conditional as defined in two-valued logical systems is in fact noted by most logicians as a 'material implication' i.e. '$p \supset q$' is defined in Russell's Principia by '$\sim p \vee q$.' That which the author refers to as an implication is in reality best stated as a tautological implication or logical implication as opposed to implication in the material sense.

The author implies a difference between a substitution and a replacement although she never makes clear the distinction and at times confuses the two. There are very essential differences between the rule of substitution and that of replacement. 1) The former allows for the replacement of single sentential variables while the latter allows for the replacement of formulae of greater syntactical complexity. 2) When applying the rule of substitution all equiform variables must be replaced by equiform expressions whereas the rule of replacement allows one to replace only a single expression in a thesis, even if a second expression, equiform with the first, occurs in that thesis.

The third section of her text entitled "Abstract Mathematical Systems" is primarily concerned with groups, rings and fields with many topics subsumed under each category, i.e. modulo systems, real and complex numbers, the LaGrange theorem with regard to the order of finite groups, etc.

One must congratulate the author on her efforts in composing a text which I believe will be of great value in teacher training courses. All those concepts necessary to the understanding of the modern mathematics curricula are well covered in a most satisfactory manner. The book is laced with suitable exercises, answers to same, and the concepts are well illustrated with appropriate diagrams and easily understood examples. Because of the general readability and clarity of approach, this book will not only serve the needs of the teacher and the teacher-in-training but also, in many areas, this text is not beyond the ken of the talented high school junior or senior.

R. H. GURLAND, C. W. Post College of Long Island University

Mathematics. Edited by Samuel Rapport and Helen Wright with foreword by Hollis R. Cooley. New York University Library of Science. Washington Square Press, New York, 1964. xii+319 pp. Paperbound $.60.

Mathematics is but one of a multivolume series of anthologies concerned with the natural and physical sciences which will comprise the New York University Library of Science. This volume offers a *pot pourri* of excerpts from mathematical literature under four general headings: "Biography of Mathematics" which offers tidbits of the lives and works of Archimedes, Newton, Gauss, Galois, Hamilton and Einstein; "Mathematics and the Mathematician" which develops some ideas basic to the nature, essence and philosophy of the mathematical enterprise through the writings of Whitehead, Hardy and Poincaré; "Some Branches of Mathematics" which encompasses cursory bits of geometry, calculus, topology, etc.; and finally "Mathematics and the World Around Us" which attempts to present mathematics as a means for the explication of physical phenomena, a tool for unravelling the mysteries of our empirical world.

The grounds upon which one can judge the merits of an anthology of such a nature are few. The strength of any anthology rests upon the power, scope and nature of the selected writings, the editorial commentary which binds these writings into a cohesive whole, and finally upon whether or not the final product satisfies or realizes the intended aim of the editors. In the case of *Mathematics* one would readily admit both the interest and value of the selected writings, however, on the second and third counts mentioned above, this reviewer has serious reservations. This book is "addressed to laymen" with the purpose of supplying mathematical literature to those general readers who desire to fill the mathematical gaps in their intellectual development. This, in turn, may be due to inadequate training and misadventures in early schooling which resulted in disenchantment and loss of confidence in their ability to handle mathematics. If such is the intent, the editors fail, due to the simple fact that the commentary necessary for the proper appreciation of the included materials, is lacking. The materials are inadequately introduced, hence continuity and purpose are not clearly established.

It is of further interest to this reviewer that the powers attributed to T. C. Mits (the common man in the street) are, many times, very inconsistent with his actual powers, that is, if he could understand all that the authors claimed he could, one would hardly tag him with the label 'layman.' A startling example of such over-estimation of the powers of the general reader is evident in the excerpt from Einstein entitled, "The $E=Mc^2$ Equation."

In closing, however, it should be noted that the articles concerned with the autobiographical sketches of men of mathematics can be appreciated by all and may well serve to stimulate the general reader to attempt the reading of E. T. Bell's fine tome, *Men of Mathematics*. For the reasonably apt and skilled person who has already attained some degree of mathematical competence, this book affords some interested moments of bedside reading.

R. H. GURLAND, C. W. Post College of Long Island University

The Teaching of Mathematics. By Roy Dubisch. Wiley, New York, 1963. 124 pp. Paperbound $2.95.

This convenient handbook for the teacher of mathematics was written with two purposes in mind: (1) "To provide the new teacher of mathematics with some general guidelines on the teaching of mathematics and some specific suggestions in regard to classroom procedures." (2) "To provide both the inexperienced and the experienced teacher with an annotated bibliography of articles on the teaching of mathematics from the intermediate algebra level through first year calculus." It is addressed to both the high school and the college teacher of mathematics at these respective levels.

In the first three chapters the author, reflecting upon his own fruitful experience in the mathematics classroom, presents his general philosophy of teaching, "the aims of teaching mathematics" and "general remarks" concerning the problems of teaching in this field. He presents the needs of the prospective mathematician, of the prospective high school teacher, of the prospective scientist and of the liberal arts student. Among other things he believes that "the primary aim of all mathematics teaching (is) the emphasis on the thinking process," that the abstract nature of mathematics should be emphasized, that there is considerable unawakened interest among students in the logic and philosophy of mathematics, and that applications of mathematics should not be neglected. He expresses regret that prospective high school teachers "often not considered as being 'real' mathematics majors" frequently "flee from the chilly halls of mathematics to the warm Embrace of Education" and suggests that "every college mathematics teacher . . . should first learn to treat these people as prospective allies in a common cause and not as second-raters," a very wise and penetrating suggestion which, if followed, could indeed make "the chilly halls of mathematics" far less "chilly."

One unique feature of this publication is the very helpful manner in which the author uses other publications to present varying practices with respect to specific teaching problems. His general procedure is to supplement his own point of view with that of other teachers by providing direct reference to alternative procedures. To this task he brings not only a wide acquaintance with mathematics texts but also a familiarity with professional literature concerning the teaching of mathematics. His bibliography which includes many well selected articles from professional periodicals should prove to be of large value to any teacher of mathematics. Its value might have been increased, however, had the few books listed which deal with the teaching of mathematics included all those which reflect the impact of recent changes in both content and emphases.

Short chapters dealing with the teaching of intermediate algebra, trigonometry and logarithms, analytic geometry, differential and integral calculus, "which constitute the great bulk of our advanced high school and elementary college teaching," will interest those concerned with the teaching of these subjects. For each such subject, the author defines selected topics which he considers significant, discusses the teaching of these topics from his own point of view, supplemented by references to periodicals and texts where alternative procedures are followed. Opposing positions on such controversial questions as the level of rigor to be used in the development of mathematical principles are included in these references which greatly enriches their value.

This little book has nothing to say to the teacher of informal or demonstrative geometry and it will be of small value to a teacher of first year algebra for, according to the author, "The problems of teaching beginning algebra are considerably different from those of advanced algebra, and we do not consider them here." Nor does the author consider explicitly the responsibility of mathematics teachers to nourish "the growth of mathematical ideas" which is a "general guideline" of great importance to both the "new" and "experienced" teacher. It is the emphasis of the Twenty-fourth Yearbook of the National Council of Teachers of Mathematics, a reference book which, if included, would have increased the value of the bibliography. The author, however, has been faithful to the two purposes which stimulated him to provide mathematics teachers with this helpful handbook and it will be a splendid addition to the professional library of all who teach "from the intermediate algebra level through first-year calculus." College teachers of mathematics, for whom professional assistance is not widely available, will find it especially valuable.

H. P. FAWCETT, Columbus, Ohio

Analytic Trigonometry. By David Luckham. Encyclopaedia Britannica Press, Chicago, 1962. 674 pp. $13.50.

Trigonometry: A Practical Course. By Norman A. Crowder and Grace C. Martin. Doubleday, Garden City, New York, 1961. 250 pp. $4.95.

These programmed texts differ radically on almost any factor a reviewer might consider.

The principal contrast is in terms of mathematical content. In *Analytic Trigonometry*, the trigonometric functions are defined as functions of a real variable, using the wrapping function approach. The principal content implied by this contemporary point of view is present: preliminaries on real numbers, set-relations-functions; definition, graphing, periodicity, and relations among the six functions; identities (developed from cos $(a+b)$) and conditional equations; inverse functions; applications to geometry of triangles and to complex numbers. In *Trigonometry: A Practical Course*, the trigonometric functions are thought of in the traditional fashion of ratios associated with a right triangle, with the resulting restriction to right and oblique triangle-solving.

Differences are sharp on several other important counts. *Analytic Trigonometry* can be used as a full course by mathematically competent students in the late-high-school or early-college years; prerequisites are two years of algebra and a year of geometry. *Trigonometry: A Practical Course* is intended as a short "course" for adults seeking enjoyment or immediate usefulness; the only prerequisites are arithmetic and a sense of algebra. The pedagogical style of the former is much more typical of a school text than the latter. *Trigonometry: A Practical Course* is in the programming mode known either by the name of one author (Crowder), or by its paging (a *scrambled* text). The other text is in the linear, or Skinnerian style.

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Should an adult want an "instant trigonometry," then *Trigonometry: A Practical Course* is appropriate. The reviewer suggests this improvement: redo this practical course in terms of the wrapping function definitions. This contemporary view has wider applications; the educated layman needs to be exposed to the fun and usefulness of that approach. The right-triangle definitions are too restrictive, with triangle-solving the only application.

ROBERT KALIN, Florida State University

ANNOUNCEMENT OF L. R. FORD AWARDS

At its meeting on January 27, 1965, in Denver, Colorado, the Board of Governors authorized a number of awards, to be named after Lester R. Ford, Sr., to authors of expository articles published in the MONTHLY and this MAGAZINE. A maximum of six awards will be made annually; each award is in the amount of $100. The articles are to be selected by a subcommittee of the Committee on Publications appointed for this purpose.

The first recipients of these awards, selected by a committee consisting of R. P. Boas, Chairman; C. W. Curtis and R. P. Dilworth, were announced by President Wilder at the Business Meeting of the Association on August 31, 1965, at Cornell University. The recipients of the Ford Awards for articles published in 1964 were the following:

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HENRY L. ALDER, *Secretary*

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PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted. Solutions should be submitted on separate, signed sheets. Send all communications for this department to Robert E. Horton, Los Angeles City College, 855 North Vermont Avenue, Los Angeles, California 90029.

PROPOSALS

593. *Proposed by J. A. H. Hunter, Toronto, Canada.*

$$\begin{array}{r} S\ E\ E \\ M\ M \\ S\ E\ E \\ M\ M \\ M\ M \\ \hline B\ E\ S\ T \end{array}$$

Each letter here stands for a particular and different digit. This alphametic expresses a simple truth, for there is nothing at all odd about our $M\ M$. So what is $B\ E\ S\ T$?

594. *Proposed by Leon Bankoff, Los Angeles, California.*

If R is the circumradius, r the inradius, and AD, BE, CF the altitudes of triangle ABC, show that

$$AD + BE + CF \leqq 2R + 5r.$$

595. *Proposed by Douglas Lind, University of Virginia.*

Form the decimal number N in the following manner: the nth digit of N is the sum of the units digit of n, the tens digit of $n+1$, the hundreds digit of $n+2$, etc. If this sum is greater than ten, the excess is carried over in the usual way. Prove that N is a repeating decimal and find its fractional equivalent.

596. *Proposed by William K. Viertel, State University Agricultural and Technical College, Canton, New York.*

Prove that there is no area bounded by a curve of the family $y=x^n$, $(n>0)$, the x-axis, and the line $x=a$, for which:

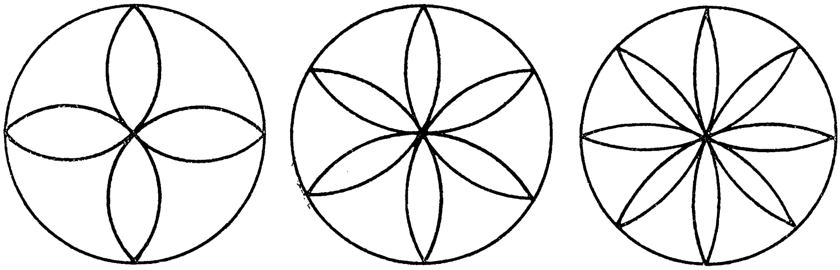
(a) the abscissa of the centroid, $\bar{x}$, is the same as the radius of gyration R_y with respect to the y-axis; or

(b) the abscissa of the centroid, $\bar{x}_a$, is the same as the radius of gyration R_a with respect to the line $x=a$; or

(c) the ordinate of the centroid, $\bar{y}$, is the same as the radius of gyration R_x with respect to the x-axis.

597. *Proposed by Alan Sutcliffe, Congleton, Cheshire, England.*

It is possible to draw regular rosettes inside a circle with any even number of leaves greater than 2. The first 3 such rosettes are shown here. What is the limit of the proportion of the area of the circle that the leaves cover as their number increases without bound?



598. *Proposed by H. Tracy Hall, Brigham Young University.*

Given a flexible, thin-walled cylinder, such as a soda straw, of diameter D. What is the edge length L of the largest regular tetrahedron that can be pushed through the straw?

599. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

If a, b, and c are any three vectors in 3-space, then show that the vectors

$$ax(bxc),\quad bx(cxa),\quad cx(axb)$$

are linearly dependent.

SOLUTIONS

LATE SOLUTIONS

William E. F. Appuhn, Long Island University: **565.** *Charles L. Belna, University of Dayton:* **561, 568.** *Martin J. Brown, Cincinnati, Ohio:* **571.** *Joy Ann Chuck, Allan Chuck, and Peter Goldstein (jointly), San Francisco, California:* **565.** *R. J. Cormier, Northern Illinois University:* **567.** *Wahin Ng, San Francisco, California:* **559.** *Wade E. Philpott, Lima, Ohio:* **565.** *Stanley Rabinowitz, Far Rockaway, New York:* **567, 569, 571.** *Donald R. Wilder, Rochester, New York:* **567.**

A Simple Graph Problem

562. [September, 1964] *Proposed by J. W. Moon and L. W. Beineke, University of London.*

A simple graph G with n points and e edges has the property that of every four points belonging to G some three of these form a triangle. How large must e be for this to be possible?

Solution by Scott B. Guthery, Miami University, Ohio.

As stated, the problem is insolvable since by C. Berge, *American Mathematical Monthly*, 71 (1964) 471–481, a graph G is simple if G "consists of two disjoint sets X and Y and a multi-valued function Γ mapping X into Y." Hence, if three points of G form a triangle, X and Y are not disjoint.

Now, if G is any graph satisfying the above-stated condition, then

$$e \geqq \frac{(n-1)(n-2)}{2}.$$

Given n points, we construct the equality case as follows: Select one point as an isolated point; connect each of the remaining $(n-1)$ points to the other $(n-2)$ points. Hence, any selection of four points must include three of which each is connected to the other, forming a triangle.

That the case is minimal is seen by considering a graph, G', of n points and $e'<e$. Evidently, there exists at least two points of G which are not connected to two other distinct points of G; hence, no three of the four form a triangle.

Also solved by Robert H. Cornell, Concord, Massachusetts; Peter M. Gibson, U. S. Naval Research Laboratory, Washington, D. C.; and the proposers. Three incorrect solutions were received.

A Memorial Cryptarithm

572. [January, 1965] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

To the memory of President Kennedy. Mr. J. F. Kennedy was killed on November 22, 1963. That is, on the day 11-22-1963. Solve the cryptarithm

$$JF\cdot(KEN + NEDY) = (11 + 22)\cdot 1963$$

in the decimal system.

Solution by Harry M. Gehman, SUNY at Buffalo, New York.

Since $(11+22)\cdot 1963$ is the product of the four primes 3, 11, 13 and 151, the only possible values of JF are 13 and 39. The latter case leads to a contradiction, and hence $J=1$ and $F=3$. From this, it follows that $KEN+NEDY=4983$, which leads to $N=4$, $Y=9$, and either $K=2$, $E=7$, $D=0$ or $K=7$, $E=2$, $D=5$. Thus

$$\begin{aligned}(11 + 22)\cdot 1963 &= 13\cdot(274 + 4709)\\ &= 13\cdot(724 + 4259).\end{aligned}$$

The fact that this problem has two solutions means (to a Republican) that JFK was *not* unique.

Also solved by Robert H. Anglin, Danville, Virginia; Merrill Barneby, University of North Dakota; Murray Berg, Standard Oil Company, San Francisco, California; Charles R. Berndtson, Institute of Naval Studies, Cambridge, Massachusetts; Dermott A. Breault, Sylvania A.R.L., Waltham, Massachusetts; Robert Brodeur, Lachine, Canada; Maxey Brooke, Sweeny, Texas; Allan Chuck, San Francisco, California; R. J. Cormier, Northern Illinois University; Monte Dernham, San Francisco, California; Herta T. Freitag, Roanoke, Virginia; Philip Fung, Fenn College, Ohio; Anton Glaser, Pennsylvania State University, Ogontz Campus; Elmer E. Hunt, Jr., Boise Junior College, Boise, Idaho; J. A. H. Hunter, Toronto, Canada; Joel V. Kamer, Cambridge, Massachusetts; John Koelzer, University of Iowa; Wahin Ng, San Francisco, California; C. C. Oursler, Southern Illinois University (Edwardsville); Harry Panish, Pomona, California; Lawrence A. Ringenberg, Eastern Illinois University; Sidney Spital, California State Polytechnic College; P. K. Subramanian, Miami University, Ohio; Charles W. Trigg, San Diego, California; William K. Viertel, State University Agricultural and Technical College, Canton, New York; Dale Woods, Northeast Missouri State Teachers College; Charles Ziegenfus, Madison College, Virginia; and the proposer.

An Aliquot Bequest

573. [January, 1965] *Proposed by Dewey Duncan, Los Angeles, California.*

A father bequeathed his herd of m horses to be divided among his sons, each son to receive an aliquot part of them. The specific divisions being impossible, a neighbor lends one horse to the herd which then made the prescribed partition possible, leaving the neighbor's horse untaken. Obtain all possible situations for:

a) Two sons; e.g., 1/2, 1/3 of herd of five horses.

b) Three sons; e.g., 1/2, 1/3, 1/7 of herd of 41 horses.

c) A solution for n sons, each son to receive a different aliquot part of the herd.

Solution by the proposer.

a) Let the two son's shares be, respectively, $1/a$, $1/b$, of herd of m horses. Accordingly, one requires all the integral solutions of the Diophantine equation

$$\frac{m+1}{a}+\frac{m+1}{b}=m,$$

or

$$(m+1)(a+b)=abm,$$

which are found to be

$$(1/a,\ 1/b,\ m)=(1/2,\ 1/3,\ 5),\ (1/3,\ 1/3,\ 2),\ (1/2,\ 1/4,\ 3),$$

and the solution (1/4, 1/4, 1) which is to be discarded as the horse is not to be shared.

b) Let the son's shares be, respectively $1/a$, $1/b$, $1/c$, of the herd m. The appropriate equation is

$$\frac{m+1}{a}+\frac{m+1}{b}+\frac{m+1}{c}=m,$$

or

$$(m+1)(ab+bc+ac)=mabc,$$

which yields the solutions,

$$\begin{aligned}(1/a,\ 1/b,\ 1/c,\ m) = {} & (1/2,\ 1/3,\ 1/4,\ 41),\\ & (1/2,\ 1/3,\ 1/9,\ 17),\ (1/2,\ 1/3,\ 1/12,\ 11),\\ & (1/2,\ 1/4,\ 1/5,\ 19),\ (1/2,\ 1/4,\ 1/6,\ 11),\\ & (1/2,\ 1/5,\ 1/5,\ 9),\ (1/2,\ 1/3,\ 1/8,\ 23),\\ & (1/2,\ 1/4,\ 1/8,\ 7),\ (1/2,\ 1/6,\ 1/6,\ 5),\\ & (1/3,\ 1/3,\ 1/4,\ 11),\ (1/3,\ 1/3,\ 1/6,\ 5),\\ & (1/4,\ 1/4,\ 1/4,\ 3).\end{aligned}$$

The other integral solutions of the foregoing equation must be discarded as they would entail sharing of single horses between two sons. They are:

$$(1/2,\ 1/8,\ 1/8,\ 3),\ (1/2,\ 1/3,\ 1/10,\ 14),$$
$$(1/2,\ 1/5,\ 1/10,\ 4),\ (1/2,\ 1/4,\ 1/12,\ 5),$$
$$(1/2,\ 1/3,\ 1/15,\ 9),\ (1/3,\ 1/4,\ 1/4,\ 5),$$
$$(1/3,\ 1/4,\ 1/6,\ 3),\ (1/3,\ 1/6,\ 1/6,\ 2),$$
$$(1/3,\ 1/4,\ 1/12,\ 2);$$

and the solution which entails sharing portions of a horse among all three sons, namely,

$$(1/4,\ 1/4,\ 1/6,\ 2).$$

c) The division of a herd of 2^n-1 horses among n sons is, obviously,

$$(1/2,\ 1/4,\ 1/8,\ \cdots,\ 1/2^n,\ 2^n-1),$$

n denoting any positive integer.

Note: If the prescribed shares are not restricted to aliquot parts, i.e., are any proper fractions rather than unit fractional parts, the sets of solutions are infinite in number, e.g.,

$$(1/2,\ k/2k+1,\ 4k+1),$$

or more generally, $(1/n,\ (n-2+nk-k)/(n-1+kn),\ n^2-n+kn^2-1)$, or most generally,

$$(p/q,\ x/y,\ qy-1),$$

wherein y denotes any solution of the congruence $py\equiv -1 \pmod{q}$, and x the corresponding solution obtained from $x=(q-p)y-1$. In the foregoing solutions n, k have any positive integral values.

Also solved by William K. Viertel, State University Agricultural and Technical College, Canton, New York. William E. F. Appuhn, Long Island University. One incorrect and one partial solution were received.

A Problem with Spheres

574. [January, 1965] *Proposed by Alan Sutcliffe, Kidsgrove, Stoke-on-Trent, Staffordshire, England.*

A bowl with a flat base is part of a sphere of internal radius 13 inches. The base of the bowl is a plane 7 inches below the center of the sphere. In it rest three smaller equal spheres, each of which touches the base of the bowl, the side of the bowl, and the other two spheres.

a) What is the radius of each of the small spheres?

b) Find complete solutions in integers for large bowl of radius R, small spheres of radius r, and base k units below the center of the sphere.

Solution by Charles W. Trigg, San Diego, California.

If the plane determined by the points of contact of the three small spheres with the large sphere is x units above the flat base, then as is evident from the figure, $R/(k-x)=r/(r-x)$ whereupon $x=r(R-k)/(R-r)$. Now the points of contact of the small sphere with the flat base are $2r/\sqrt{3}$ units from the diameter of the large sphere perpendicular to the base. So

$$\sqrt{(r^2-(r-x)^2)}+2r/\sqrt{3}=\sqrt{(R^2-(k-x)^2)}.$$

When the value of x is substituted in this equation, simplification gives

$$2r/\sqrt{3}=\sqrt{((R-k)(R+k-2r))}.$$

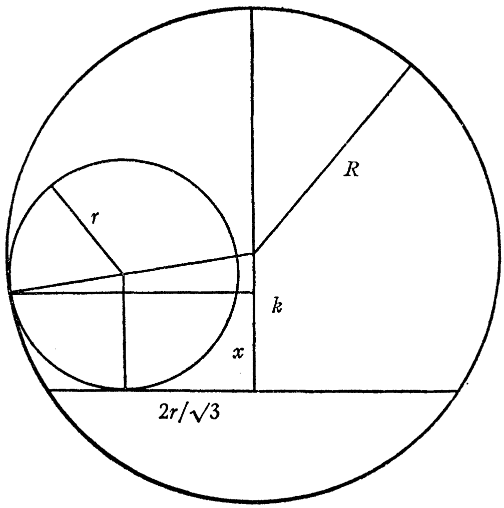
That is,

$$4r^2+6(R-k)r-3(R^2-k^2)=0.$$

So,

$$r=[-3(R-k)+\sqrt{(3(R-k)(7R+k))}]/4.$$

In this problem, $R=13''$, $k=7''$, so $r=6''$.



A one-parameter solution is

$$R=n^2+3n+3, \qquad k=n^2+3n-3, \qquad r=3n, n=1,2,3\cdots.$$

Also solved by Leon Bankoff, Los Angeles, California; Ned Harrell, Menlo-Atherton High School Atherton, California; and the proposer.

Expectation by Difference Calculus

575. [January, 1965] *Proposed by Fred Terry, Northwestern University*

Objects labeled 1, 2, $\cdots$, 10 are arranged in a line from left to right in random order. 1 through 4 are red, and 5 through 10 are black. Let x equal the position number, counting from the left, in which a red object first appears.

a) Find $E(x)$ when drawing without replacement.
b) Solve by Difference Calculus only.

Solution by Sidney Spital and Cameron Bogue (jointly), California State Polytechnic College, Pomona, California.

We shall solve both parts of the problem for a general number of objects n, r of which are red and b black.

(a) Denoting as p_i the probability that the first red appears at the ith position,

$$p_i = \frac{\binom{b}{i-1}\binom{n}{1}}{\binom{n}{i}} \qquad (i = 1, 2, \cdots, b+1).$$

With some algebra and use of $n=r+b$, we can rewrite

$$p_i = \frac{\binom{n-i}{r-1}}{\binom{n}{r}}.$$

Hence, the expected position of the first red is given by

$$E(x) = \frac{1}{\binom{n}{r}} \sum_{i=1}^{b+1} \binom{n-i}{r-1} i$$

$$= \frac{1}{\binom{n}{r}} \sum_{i=1}^{b+1} \binom{n-i}{r-1} (n+1-(n-i+1))$$

$$E(x) = \frac{1}{\binom{n}{r}} \left[(n+1) \sum_{i=1}^{b+1} \binom{n-i}{r-1} - r \sum_{i=1}^{b+1} \binom{n-i+1}{r} \right].$$

Both terms are summable via the identity in binomial coefficients

$$\sum_{j=k}^{p} \binom{j}{k} = \binom{p+1}{k+1}$$

giving

$$E(x) = \frac{1}{\binom{n}{r}} \left[(n+1)\binom{n}{r} - r\binom{n+1}{r+1} \right]$$

from which, $E(x)=(n+1)/(r+1)$.

(b) For the difference calculus method, it is necessary to recognize our increased generality by relabelling $E(x)=m(r, b)$. We construct a recurrence relation for $m(r, b)$ by making the first draw and using conditional expectations. Respectively calling R and B the events of red and black first draws, we have $m(r, b)=m(r, b|R)\cdot p(R)+m(r, b|B)\cdot p(B)$. Since $m(r, b|R)=1$ and $m(r, b|B)=1+m(r, b-1)$, it follows

$$m(r, b) = \frac{r}{r+b} + \frac{b}{r+b}(1 + m(r, b-1))$$

$$m(r, b) - \frac{r}{r+b} m(r, b-1) = 1$$

which is a first order linear (variable coefficient) difference equation for the sequence $\{m(r, 0), m(r, 1), m(r, 2), \cdots\}$ with the easily seen starting value $m(r, 0)=1$. The solution of this equation is known to be (see, for example, P. Henrici, *Elements of Numerical Analysis*, Wiley, New York, 1964, p. 50):

$$m(r, b) = \frac{1}{\binom{r+b}{b}} \sum_{i=0}^{b} \binom{r+i}{i}.$$

It then follows from another identity in binomial coefficients

$$m(r, b) = \frac{1}{\binom{r+b}{b}} \binom{r+b+1}{b} = \frac{r+b+1}{r+1} = \frac{n+1}{r+1}.$$

Hence, both parts are in agreement and for the numerical values given by the problem

$$E(x) = m(4, 6) = \frac{10+1}{4+1} = \frac{11}{5}.$$

Also solved by Joseph Lev, New York State Education Department, Albany, New York; and the proposer.

The Diagonals of a Quadrangle

576. [January, 1965] *Proposed by V. F. Ivanoff, San Carlos, California.*

A quadrangle with area Q is divided by its diagonals into four triangles with areas A, B, C, and D. Show that

$$A\cdot B\cdot C\cdot D = \frac{(A+B)^2(B+C)^2(C+D)^2(D+A)^2}{Q^4}.$$

I Solution by Stephen Hoffman, Trinity College, Connecticut.

We let the vertices of the quadrangle be P, Q, R, S, and the intersection of the diagonals be T. We now let $a=\overrightarrow{PQ}$, $b=\overrightarrow{QR}$, $c=\overrightarrow{RS}$, and let k be the scalar $(0<k<1)$ such that $\overrightarrow{PT}=k(a+b)$.

For the area, we have

$$A=\frac{k}{2}\,|a\times b|,\qquad B=\frac{1-k}{2}\,|a\times b|,$$

$$C=\frac{1-k}{2}\,|c\times(a+b)|,$$

and

$$D=\frac{k}{2}\,|c\times(a+b)|.$$

Thus $Q=\frac{1}{2}[|a\times b|+|c\times(a+b)|]$, and

$$ABCDQ^4=\frac{k^2(1-k)^2}{256}\,|a\times b|^2|c\times(a+b)|^2[|a+b|+|c\times(a+b)|]^2.$$

On the other hand,

$$A+B=\frac{1}{2}\,|a\times b|,$$

$$B+C=\frac{1-k}{2}\,[|a\times b|+|c\times(a+b)|],$$

$$C+D=\frac{1}{2}\,|c\times(a+b)|,$$

and

$$D+A=\frac{k}{2}\,[|a\times b|+|c\times(a+b)|],$$

so that

$$(A+B)^2(B+C)^2(C+D)^2(D+A)^2=ABCDQ^4.$$

II *Solution by Leon Bankoff, Los Angeles, California.*

If the four triangles are named in cyclic order, we have

$$\frac{A}{A+B}=\frac{A+D}{Q};\quad \frac{B}{B+C}=\frac{A+B}{Q};\quad \frac{C}{C+D}+\frac{B+C}{Q};\quad \frac{D}{D+A}=\frac{C+D}{Q},$$

and

$$A\cdot B\cdot C\cdot D=\frac{(A+B)^2(B+C)^2(C+D)^2(D+A)^2}{Q^4}.$$

Also solved by Robert Brodeur, Lachine, Canada; L. Carlitz, Duke University; R. J. Cormier, Northern Illinois University; Vivian Johnson, Western Michigan University; John Koelzer and

Clifford Kottman (jointly), University of Iowa; Harry Panish, Pomona, California; Stanley Rabinowitz, Far Rockaway, New York; J. P. Ruebsamien, SUNY at New Paltz, New York; Sidney Spital, California State Polytechnic College; Charles W. Trigg, San Diego, California; John Weddington, Levack District High School, Levack, Canada; Dale Woods, Northeast Missouri State Teachers College; and the proposer.

An Inequality in Powers

577. [January, 1965] *Proposed by Murray S. Klamkin, SUNY at Buffalo, New York.*

Show that if $x_n \geqq x_{n-1} \geqq \cdots \geqq x_2 \geqq x_1 \geqq 0$, then $x_2^{x_2} x_2^{x_3} \cdots x_n^{x_1} \geqq x_2^{x_1} x_3^{x_2} \cdots x_1^{x_n}$ for $n \geqq 3$, with equality holding only if $n-1$ of the numbers are equal.

Solution by L. Carlitz, Duke University.

We may assume that $x_1 > 0$. Then the stated inequality is equivalent to

$$\left(\frac{x_2}{x_1}\right)^{x_3} \cdots \left(\frac{x_n}{x_1}\right)^{x_1} \geqq \left(\frac{x_2}{x_1}\right)^{x_1} \cdots \left(\frac{x_n}{x_1}\right)^{x_{n-1}}.$$

We may therefore assume that $x_n \geqq \cdots \geqq x_2 \geqq x_1 = 1$.

For $n=3$ put $x_2 = 1+a$, $x_3 = 1+b$, where $b \geqq a$. The stated inequality becomes $(1+a)^{1+b}(1+b) \geqq (1+a)(1+b)^{1+a}$, that is,

$$(1) \qquad (1+a)^b \geqq (1+b)^a.$$

This is an immediate consequence of Bernoulli's inequality. Moreover, we have equality if and only if $a=b$ or $a=0$.

In the general case, we wish to show that

$$\prod_{s=2}^{n-1} x_s^{x_{s+1}} \cdot x_n \geqq x_2 \prod_{s=3}^{n} x_s^{x_{s-1}}.$$

If we put $x_s = 1 + a_s$, $\frac{1}{2} \leqq s \leqq n$, this inequality becomes

$$(2) \qquad \prod_{s=2}^{n-1} (1+a_s)^{1+a_{s+1}} \cdot (1+a_n) \geqq (1+a_2) \prod_{s=3}^{n} (1+a_s)^{1+a_{s-1}},$$

where $a_n \geqq a_{n-1} \geqq \cdots \geqq a_2 \geqq 0$. Then by (1), the left member of (2) is greater than or equal to

$$\prod_{s=2}^{n-1} (1+a_{s+1})^{a_s} \cdot \prod_{s=2}^{n} (1+a_s) = \prod_{s=3}^{n} (1+a_s)^{a_{s-1}} \cdot \prod_{s=2}^{n} (1+a_s)$$

$$= (1+a_2) \prod_{s=3}^{n} (1+a_s)^{1+a_{s-1}}.$$

This proves (2).

The condition for equality in (1) is either $a=b$ or $a=0$. Thus the condition for equality in (2) is either $a_s = a_{s+1}$ or $a_s = 0$, $(s = 2, \cdots, n-1)$. Assume that

$$(3) \qquad a_2 = \cdots = a_k = 0 < a_{k+1} = \cdots = a_n;$$

then (2) becomes

$$(1 + a_n)^{(n-k-1)(1+a_n)+1} = (1 + a_2)(1 + a_n)^{(n-k-1)(1+a_n)+1},$$

provided $2 \leqq k < n$. This gives $a_n = a_2$, which contradicts (3). Hence either $a_2 = \cdots = a_{n-1} = 0$ or $a_2 = \cdots = a_{n-1} = a_n$.

Also solved by the proposer.

A Subgroup

578. [January, 1965] *Proposed by Michael Gemignani, University of Notre Dame, Indiana.*

Let G be any group, and let $C = g_1A \cap g_2B$ where A and B are subgroups of G and $g_1, g_2 \in G$. Prove there is a subgroup $D \subseteq G$ and $g_3 \in G$ such that $C = g_3D$.

Solution by Steve Ligh, University of Kentucky.

Assume C is nonempty. If $x \in C$, then $x = g_1a = g_2b$, $a \in A$, $b \in B$; and it is easy to show that $a_i^{-1}a_j = b_i^{-1}b_j$ for all $a_i \in A$, $b_i \in B$ such that $g_1a_i = g_2b_i$ for all i and $g_1a_i \in C$.

Now define

$$D = (d \mid d = a_0^{-1}a,\ a_0,\ a \in A,\ g_1a_0,\ g_1a \in C,\ a_0 \text{ fixed}),$$

and we want to show D is a subgroup of G. Let $d_1 = a_o^{-1}a_1$, $d_2 = a_o^{-1}a_2$, then

$$d_1d_2^{-1} = a_0^{-1}a_1a_2^{-1}a_0,$$

so that we want to show $g_1a_1a_2^{-1}a_0 \in C$.

Suppose $g_1a_1a_2^{-1}a_0 \notin C$, then $g_1a_1a_2^{-1}a_0 = g_2b_1b_2^{-1}b_0 \notin g_2B$, which means that $b_1b_2^{-1}b_0 \notin B$. But this is a contradiction.

Finally, let $g_3 = g_1a_0$ and it is clear that $g_3D = C$.

Also solved by John L. Brown, Jr., Ordnance Research Laboratory, State College, Pennsylvania; Dennis P. Geller, Harpur College, New York; Stephen Hoffman, Trinity College, Connecticut; A. M. Vaidya, Texas Technological College; John Weddington, Levack District High School, Levack, Canada; and the proposer.

Heterosquares

84. [January, 1951] *Proposed by Dewey Duncan, Los Angeles, California.*

We define a heterosquare as a square array of the first n^2 positive integers, so arranged that no two of the rows, columns and diagonals (broken as well as straight) have the same sum.

a) Show that no heterosquare of order two exists.

b) Find a heterosquare of order three.

Solution by Charles F. Pinzka, University of Cincinnati.

We assert that there are no heterosquares of order <4.

(a) The case $n = 2$ is obvious, since the sums $1+4=5$ and $2+3=5$ will always arise.

(b) We first note that the sums which arise are invariant under row and column transpositions, diagonal reflections, and the shearing transformation

$$\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix} \rightarrow \begin{matrix} a & b & c \\ f & d & e \\ h & i & g \end{matrix}$$

Allowing for these transformations, the first four digits can be assigned in the following seven essentially different ways:

(1) $\begin{matrix} 1 & 2 & 3 \\ 4 & e & f \\ g & h & i \end{matrix}$ (2) $\begin{matrix} 1 & 2 & 4 \\ 3 & e & f \\ g & h & i \end{matrix}$ (3) $\begin{matrix} 1 & 2 & c \\ 3 & 4 & f \\ g & h & i \end{matrix}$ (4) $\begin{matrix} 1 & 2 & c \\ 3 & e & 4 \\ g & h & i \end{matrix}$

(5) $\begin{matrix} 1 & 2 & c \\ 3 & e & f \\ 4 & h & i \end{matrix}$ (6) $\begin{matrix} 1 & 2 & c \\ 3 & e & f \\ g & 4 & i \end{matrix}$ (7) $\begin{matrix} 1 & 2 & c \\ 3 & e & f \\ g & h & 4 \end{matrix}$

Since in each case the remaining digits may be assigned in $5!=120$ ways, there are at most 840 basic configurations. (There are even fewer basic configurations than this if we eliminate complementary configurations, constructed as follows: For a given configuration, replace each element x by $10-x$; each sum S will be replaced by $30-S$ in the new configuration and heterosquares will remain heterosquares. However, not all complementary configurations will be distinct from their originals.)

Case (3) is disposed of immediately, since $1+4+i=2+3+i$. The other cases are somewhat tedious, the analysis for Case (1) being typical:

6, $g+5$, $h+7$, $i+6$, $e+f+4, \cdots, g+h+i$ must be distinct, leading to: $e\neq 5$ or $f+2$; $f\neq 6$ or $e+1$; $g\neq e+2, f+1, h+2$, or $i+1$; $h\neq e+3, f+2, g+1$, or $i+2$; $i\neq e+1, f+3, h+1$, or $g+2$. The inequalities involving e or f only reduce the possibilities for (e, f) to (6, 5), (6, 8), (6, 9), (7, 9), (8, 5), (8, 7), (9, 5), and (9, 8). With the remaining inequalities, a table of possibilities may be constructed as follows:

e	f	g	h	i
6	5	7	8	9
6	8	5, 7	5, 7	9
6	9	5, 7	5, 7, 8	5, 8
7	9	5, 6, 8	5, 6, 8	5, 6
8	5	7, 9	6, 9	6, 7
8	7	9	5, 6	5, 6
9	5	7, 8	6, 8	6, 7
9	8	5, 6, 7	5, 6, 7	5, 6, 7

Continued application of the inequalities, the process sometimes being facilitated by breaking the cases down further, soon reduces the possibilities to a number small enough to be checked directly. In all of the original seven cases, there were found to be no heterosquares.

We were not able to find any "almost" heterosquares, that is, arrays in which all sums but two are distinct. However, there are many "near" heterosquares, that is, arrays in which all sums but three are distinct. An example is

1	2	4
3	8	7
5	9	6,

in which the sum 17 occurs three times.

Order 4 heterosquares exist. The following two examples were obtained by bordering the "near" heterosquares of order 3 given above:

1	2	4	13		1	2	4	13
3	8	7	11		3	8	7	11
5	9	6	14		5	9	6	15
15	12	16	10		14	12	16	10.

QUICKIES

Q364. Points E, F are taken on sides BC and AD, respectively, of parallelogram $ABCD$. AE intersects BF at G and ED intersects CF at H. Prove that GH prolonged bisects the parallelogram.

[*Submitted by Mannis Charosh*]

Q365. If 70% of the adult males in a community have brown eyes, 75% have dark hair, 85% are over 5 ft. 8 in. tall, and 90% weigh more than 140 lbs., what percent at least have all four characteristics?

[*Submitted by C. W. Trigg*]

Q366. Solve

$$x + y + z = 3$$
$$x^2 + y^2 + z^2 = 7/2$$
$$x^3 + y^3 + z^3 = 9/2$$

[*Submitted by M. S. Klamkin*]

Q367. Prove that at least two people have the same number of friends.

[*Submittted by D. L. Silverman*]

Q368. The rows and columns of the multiplication table of a group

$$G = \{a, b, c, d, e, f, g\}$$

are headed by the elements in this order. The first five entries in the second row are b, d, f, c, a. Complete the multiplication table.

[*Submitted by Richard Laatsch*]

Answers on page 211

without using the notion of "Möbius function."

This proof is really very simple when presented. But this is the beauty of Number Theory. As once Gauss wrote in one of his letters to a talented and learned lady, Sophie Germain: "Les charmes enchanteurs de cette science sublime ne se décèlent dans toute leur beauté qu'à ceux qui ont le courage de l'approfondir."

Summer Research Institute, The Canadian Mathematical Congress, Vancouver, 1964.

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ANSWERS

A364. If AC and BD intersect at I, then G, H, and I are collinear by Pappus' Theorem. Any line through I bisects the parallelogram.

A365. $(70+75+85+90)-300$ or 20%. In general,

$$P = \sum_{i=1}^{n} p_i - 100(n-1).$$

A366. Let x, y, z be the roots of $s^3+a_1s^2+a_2s+a_3=0$. Then

$$a_1 = 3, \qquad \sum x^2 = \left(\sum x\right)^2 - 2\sum xy \quad \text{and} \quad a_2 = 11/4.$$

Next

$$\sum x^3 + a_1\sum x^2 + a_2\sum x + 3a_3 = 0 \quad \text{and} \quad a_3 = -3/4.$$

The roots of the cubic are 1/2, 2/2, and 3/2.

A367. Suppose otherwise. If the world population is n, this implies that for each of the numbers $0, 1, 2, \cdots, n-1$, one person has that number of friends. But if one person is a friend of everyone else, no one is friendless. Thus the 0 and $n-1$ are incompatible.

A368. The equation $ba=b$ implies that a is the identity. The group has order 7, is therefore cyclic and generated by b. The given information yields $b^2=d$, $b^3=bd=c$, and $b^4=bc=f$. Then $ag=g$ implies that $bg\neq g$, thus $b^5=bf=g$ and $b^6=e$. The table follows from the laws of exponents.

(Quickies on page 252)